

Instantons, black holes, and harmonic functions

Thomas Mohaupt

*Theoretical Physics Division, Department of Mathematical Sciences, University of
Liverpool, Peach Street, Liverpool L69 7ZL, United Kingdom*
E-Mail: Thomas.Mohaupt@liv.ac.uk

Kirk Waite

*Theoretical Physics Division, Department of Mathematical Sciences, University of
Liverpool, Peach Street, Liverpool L69 7ZL, United Kingdom*
E-Mail: Kirk.Waite@liv.ac.uk

ABSTRACT: We find a class of five-dimensional Einstein-Maxwell type Lagrangians which contains the bosonic Lagrangians of vector multiplets as a subclass, and preserves some features of supersymmetry, namely the existence of multi-centered black hole solutions and of attractor equations. Solutions can be expressed in terms of harmonic functions through a set of algebraic equations. The geometry underlying these Lagrangians is characterised by the existence of a Hesse potential and generalizes the very special real geometry of vector multiplets.

Our construction proceeds by first obtaining instanton solutions for a class of four-dimensional Euclidean sigma models, which includes those occurring for four-dimensional Euclidean $N = 2$ vector multiplets as a subclass. For solutions taking values in a completely isotropic submanifold of the target space, we show that the solution can be expressed in terms of harmonic functions if an integrability condition is met. This condition can either be solved by imposing that the solution depends on a single coordinate, or by imposing that the target space is a para-Kähler manifold which can be obtained from a real Hessian manifold by a generalized r -map. In the latter case one obtains multi-centered solutions. Moreover, if the integrability condition is met, the second order equations of motion can always be reduced to first order equations, which become gradient flow equations if the solution is further required to depend on one coordinate only. The dualization of axions into tensor fields and the lifting of four-dimensional instantons to five-dimensional solitons are used to motivate the addition of a boundary term to the action, which accounts for the instanton action. If the sigma model is coupled to gravity, and if the Hesse potential is of a suitable form which we specify, then the four-dimensional Euclidean Lagrangian can be lifted consistently to a five-dimensional Einstein-Maxwell type Lagrangian. Instanton solutions lift to extremal black hole solutions, and the instanton action equals the ADM mass.

KEYWORDS: black holes, instantons, harmonic functions, special geometry.

Contents

1. Introduction and Overview	2
1.1 Introduction	2
1.2 Overview	6
2. Sigma models with para-Hermitian target spaces	9
2.1 Motivation and discussion of the Euclidean action	9
2.2 From harmonic maps to harmonic functions	11
2.3 From instanton charges to harmonic functions	17
2.4 Spherically symmetric solutions and flow equations	20
3. The dual picture	24
4. Finiteness of the instanton action and attractor behaviour. Examples	28
4.1 Hesse potential $\mathcal{V} = \sigma^p$	28
4.2 General homogeneous Hesse potentials	29
4.3 Hesse potential $\mathcal{V} = \frac{1}{6}C_{IJK}\sigma^I\sigma^J\sigma^K$	31
4.4 Hesse potential $\mathcal{V} = \frac{1}{4!}C_{IJKL}\sigma^I\sigma^J\sigma^K\sigma^L$	32
4.5 Hesse potential $\mathcal{V} = -\log(\sigma)$	33
4.6 Hesse potential $\mathcal{V} = -\log(\sigma^1\sigma^2\sigma^3)$	34
4.7 Hesse potential $\mathcal{V} = -\log \hat{\mathcal{V}}(\sigma)$, with homogeneous $\hat{\mathcal{V}}(\sigma)$	34
5. Lifting to five dimensions, without gravity	35
6. Lifting to five dimensions, with gravity	37
6.1 Dimensional lifting and dimensional reduction	37
6.2 ADM mass and instanton action	42
6.3 Black hole entropy and the size of the throat	43
6.4 Attractor behaviour and examples	44
6.4.1 Prepotential $\hat{\mathcal{V}}(\sigma) = \sigma^1\sigma^2\sigma^3$	45
6.4.2 Prepotential $\hat{\mathcal{V}}(\sigma) = \sigma^1\sigma^2\sigma^3\sigma^4$	46
6.4.3 General homogeneous prepotential $\hat{\mathcal{V}}(\sigma)$	46
7. Conclusions and Outlook	48
A. Complexification of the target space	50
Bibliography	57

1. Introduction and Overview

1.1 Introduction

Stationary solutions of supergravity theories, such as black holes, black p -branes, gravitational waves and Kaluza-Klein monopoles can often be presented in terms of harmonic functions, which only depend on the coordinates transverse to the worldvolume.¹ This is related to the existence of multi-centered solutions: if the field equations can be reduced to a set of decoupled harmonic equations without assuming spherical symmetry, then not only single-centered harmonic functions,

$$H(r) = h + \frac{q}{r^{D-3}} ,$$

but also multi-centered harmonic functions

$$H(\vec{x}) = h + \sum_{i=1}^N \frac{q_i}{|\vec{x} - \vec{x}_i|^{D-3}} .$$

provide solutions of the field equations. The existence of stationary multi-centered solutions requires the exact cancellation of the forces between the constituents at arbitrary distance, the classical examples being multi-centered extremal black hole solutions like the Majumdar-Papapetrou solutions of Einstein-Maxwell theory [2]. This cancellation is often explained by supersymmetry: if the theory allows an embedding into a supersymmetric theory, then one can look for solutions admitting Killing spinors, which in turn leads to stationary multi-centered solutions. The saturation of an extremality bound, which is needed for the cancellation of forces is then equivalent to the saturation of the supersymmetric mass bound (also called the BPS mass bound). The extremal Reissner-Nordström black hole, and its multi-centered generalizations are the prototypical examples of such supersymmetric solitons [3, 4]. In supergravity theories with $N \geq 2$ supersymmetry the asymptotic behaviour of BPS solutions at event horizons is determined by the charges through the black hole attractor mechanism [5, 6, 7], which forces the scalar fields to take fixed point values. The attractor mechanism and the construction of solutions in terms of harmonic functions are closely related: from the attractor equations (also called stabilisation equations or fixed point equations), which determine the asymptotic near-horizon solution one can obtain the so-called generalized stabilisation equations, which allow to express the complete solution algebraically in terms of harmonic functions [5, 7, 8, 9, 10, 11, 12, 13]. In the single-centered case the generalized stabilisation equations can be formulated equivalently as gradient flow equations for the scalars as functions of the radial coordinate [14, 15, 16]. The potential driving the flow is the central charge.

While imposing supersymmetry is sufficient to derive the attractor mechanism, and to obtain multi-centered solutions, it is not necessary. As already observed in [14] the attractor mechanism is a general feature of extremal black holes in Einstein-Maxwell type theories. More recently, single-centered non-supersymmetric extremal solutions have been

¹There are many excellent reviews on the subject, including [1], which discusses the approach we are going to use in its Section 9.

studied extensively and from various perspectives starting from [17, 18, 19]. Reviews of this subject can be found in [20, 21]. By imposing that the solution is spherically symmetric in addition to stationary, one can reduce the problem of solving the equations of motion to a one-dimensional problem which only involves the radial coordinate [14]. The reduced problem is formally equivalent to the motion of a particle on a curved target space in presence of a potential, usually called the black hole potential. One contribution to the potential depends on the charges and is obtained by eliminating the gauge fields through their equations of motion. Alternatively, one can often convert the gauge fields (or at least the those components of the gauge fields relevant for the solution) into scalars. Then the equations of motion take the form of a geodesic equation (without potential) on an extended scalar manifold which encodes all relevant degrees of freedom. The black hole potential receives further contributions if the full higher-dimensional solution involves rotation, if the gauge fields cannot be expressed in terms of scalars, and if a cosmological constant, higher curvature terms, Taub-NUT charge, or other such complications are present. We will restrict ourselves to situations which can be formulated as geodesic motion (without potential) on an enlarged scalar manifold. In this set-up one is left with solving the scalar equations of motions, while the Einstein equations themselves result in a constraint, which can be interpreted as the conservation of the particle’s energy.

The standard approach to single-centered solutions is to try rewriting the second order scalar equations of motion as first order gradient flow equations. While for BPS solutions the potential driving the gradient flow is the central charge [14], first order rewritings have since then been found for various non-BPS solutions, and the function driving the flow is referred to as the (‘fake’-, ‘generalized’ or ‘pseudo’-)superpotential or as the prepotential [22, 23, 24, 27]. The problem of finding a first order gradient flow prescription can be reformulated using the Hamilton-Jacobi formalism as the problem of finding a canonical transformation [27]. In many cases the first order equations can be interpreted as generalized Killing spinor equations, by defining a suitable covariant derivative for spinors. Similar observation have been made before in the context of cosmological solutions and domain walls, and this has motivated the concept of ‘fake’- or ‘pseudo’-supersymmetry [28, 29].

While much is known about the attractor mechanism for non-BPS black holes, we are not aware of a systematic analysis of the conditions which allow multi-centered solutions. From the supersymmetric case one is used to the observation that the existence of a first order rewriting and the reduction of the equations of motion to algebraic relations between the scalars and a set of harmonic functions (i.e. generalized stabilization equations) are closely related. In the single-centered case one might regard the generalized stabilisation equations as ‘solutions’ of the gradient flow equation.² But if one looks beyond single-centered solutions, it becomes clear that the generalized stabilization equations are much more than mere solutions to the gradient flow equations. In the absence of spherical symmetry, the gradient flow equations are replaced by first order partial differential equations, which have not been studied much in the literature. In contrast the form of the generalized stabilization equations remains the same, and non-spherical solutions simply correspond

²This ‘solution’ is in general not completely explicit, since generically one cannot find explicit expression for the scalars in terms of the harmonic functions.

to a more general choice of harmonic functions, namely multi-centered instead of single-centered harmonic functions. Also note that for BPS solutions the generalized stabilization equations can be derived directly, without passing through an intermediate stage of first finding a first order rewriting and then solving the flow equations [10, 13]. This suggests to develop an approach to non-BPS black holes which is not based on first order rewriting and flow equations, but on generalised stabilisation equations. In other words one should try to reduce the second order equations of motion to decoupled harmonic equations, without imposing spherical symmetry. We do not expect that this strategy can work for general Einstein-Maxwell theories. As remarked in [14], it is virtually impossible to obtain detailed information about the behaviour of extremal black hole solution away from the horizon and infinity if the scalar manifold is generic. The special geometries of vector multiplet manifolds provide examples where explicit solutions can be found (in general up to algebraic equations). Our strategy will be to start with general Einstein-Maxwell type Lagrangians, which include those of $N = 2$ vector multiplets as a subclass, and then to work out the additional constraints that we need to impose on the scalar manifold in order to obtain multi-centered solutions. Since we are only interested in stationary solutions, we can perform a dimensional reduction over time and work with the resulting Euclidean theory. Dimensional reduction over time is a powerful solution generating technique, which was (to our knowledge) first used in [30] and [31], and which has recently been used to explore non-BPS extremal black holes (albeit only for single-centered solutions) [22, 23, 33, 34, 24, 25, 27]. We refer to [1, 32, 38] for reviews of this method. The essential part of the reduced Lagrangian is a sigma model, whose equation of motion is the equation for a harmonic map from the reduced space-‘time’ to the scalar target space. The problem which we will investigate in this paper is to reduce the non-linear second order partial differential equation of a harmonic map to decoupled linear harmonic equations. As we will see this reduction imposes non-trivial conditions on the scalar manifold, which define a generalized version of the special geometry of $N = 2$ vector multiplets, and which is characterised by the existence of a potential for the metric.

While previous studies have either investigated supergravity Lagrangians, or Lagrangians with generic scalar manifolds, we have identified an interesting intermediate class of scalar manifolds: they are much more general as the target spaces of supergravity theories, while still allowing to express the solution in terms of harmonic functions and thus to obtain multi-centered solutions. This class of scalar manifolds is much more generic than symmetric spaces. For symmetric spaces, powerful methods from the theory of Lie groups are available, and the construction of BPS and non-BPS extremal solutions can be related to integrable systems [33, 34, 24, 25, 26, 27]. While this class of models is very interesting, symmetric spaces are not even general enough to cover the scalar geometries supergravity with $N \leq 2$ supersymmetry. Thus one is limited to models with $N > 2$ supersymmetry, or to special $N \leq 2$ models, like toroidal and orbifold compactifications, or consistent truncations of models with $N > 2$ supersymmetry.

While our analysis could (and ultimately should) be carried out in an arbitrary number of dimensions, we will be more specific and fix the number of dimensions to be five. Since our approach is guided by results on $N = 2$ vector multiplets, this is a natural choice,

because the so-called very special geometry of five-dimensional vector multiplets [35] is the simplest of the special geometries of $N = 2$ supermultiplets. From our results it will be clear that there is a similar story for four-dimensional $N = 2$ vector multiplets, but the five-dimensional case is a more convenient starting point for technical simplicity. Thus, we will start with generic five-dimensional Einstein-Maxwell theories and construct asymptotically flat, electrically charged, extremal, multi-centered solutions by using the associated four-dimensional Euclidean sigma models. By imposing that the equations of motion reduce to decoupled harmonic equations, we obtain a constraint on the scalar metric which generalizes the very special real geometry of five-dimensional vector multiplets [35]. There are in fact two relevant conditions. An integrability condition for the solution implies the existence of a Hesse potential for the scalar metric, while the consistent lifting of the four-dimensional Euclidean solution to a solution of the five-dimensional Einstein-Maxwell theory requires in addition that the Hesse potential is the logarithm of a homogeneous function, which we call the prepotential. Five-dimensional supergravity corresponds to the special case where this prepotential is homogeneous of degree three. When expressed in terms of five-dimensional variables, the algebraic relations which express the solution in terms of harmonic functions take the form of the generalized stabilization equations for five-dimensional vector multiplets [11, 12]. Therefore the solutions contain the static (non-rotating) electric multi-centered BPS solutions of five-dimensional supergravity [11, 12] as a subclass. We also consider the case where we lift solutions of a four-dimensional Euclidean sigma model without coupling to gravity. In this case the Hesse potential is not constrained, and we obtain solitonic solutions of a five-dimensional gauge theory coupled to scalars.

The construction of solutions is presented from the reduced, four-dimensional perspective, i.e. we first construct solutions of four-dimensional Euclidean sigma models and discuss the lifting to five dimensions in a second step. The target geometries of the four-dimensional sigma models include those of four-dimensional Euclidean vector multiplets, both rigid and local, as special cases. Therefore there is some overlap between this paper and work on Euclidean special geometry [36, 37, 38]. In particular, the target space geometry of the four-dimensional Euclidean sigma model is para-complex,³ and the integrability condition guaranteeing the existence of multi-centered solutions implies that it is para-Kähler. The relation between the real, Hessian target spaces of five-dimensional sigma models and the para-Kähler geometry of four-dimensional sigma models provides a ‘para-version’ or ‘temporal version’ of the generalized r -map described recently in [39]. To be precise, we find two different generalized para- r -maps, depending on whether we couple the Euclidean sigma model to gravity before lifting, or not. In [39] only the case without gravity was considered.

The solutions of the reduced Euclidean theory can be interpreted as instantons and are interesting in their own right. They are of the same type as the D-instanton solution of type-IIB supergravity [41], and the instanton solutions of $N = 2$ hypermultiplets [42, 43, 44, 45, 46, 25], and they contain the instanton solutions and $N = 2$ vector multiplets

³Para-complex geometry is explained in some detail in [36, 38]. The features relevant for our work will be explained in due course.

[38, 48] as a subclass. Since the instantons satisfy a Bogomol’nyi bound and lift to extremal black holes, we refer to them as extremal instanton solutions. Extremality is equivalent to satisfying what we call the ‘extremal instanton ansatz,’ which restricts the scalars to vary along totally isotropic submanifolds of the scalar target. This in turn is equivalent to the vanishing of the energy momentum tensor, which makes it consistent to solve the reduced Einstein equations by taking the four-dimensional reduced metric to be flat. The dimensional lifting to five dimensions then gives rise to extremal black hole solutions.

For single-centered extremal solutions of supergravity theories the distinction between BPS solutions and non-BPS solutions manifests itself in the form of the potential which drives the gradient flow equations. For BPS solutions this potential is the central charge, while for non-BPS solution it is another function, which one needs to construct. In our framework this distinction finds a geometric interpretation in terms of the para-Kähler geometry of the target space of the Euclidean sigma model, because the extremal instanton ansatz comes in two versions. The first version, which can be imposed without further constraints on the scalar metric requires that the scalar fields vary along the eigendistributions of the para-complex structure. However, if the metric has discrete isometries, a generalized version of the extremal instanton ansatz is possible, which allows the scalars to vary along other completely isotropic submanifolds of the target. This distinction generalizes the one between BPS and non-BPS extremal solutions in supergravity, and also provides a geometric interpretation of the difference between the two types of extremal instanton solutions.

Besides serving as generating solutions for higher-dimensional solitons, instanton solutions are relevant for computing instanton corrections to quantum amplitudes and effective actions. While this second application is not our main focus in this paper, we encounter one notorious problem arising in this context: if one computes the instanton action by substituting the instanton solution into the Euclidean action one obtains zero instead of the expected non-vanishing finite result. We review one of the proposed solutions, namely the dualization of axions into tensor fields [41]. In this dual picture the Euclidean action is positive definite and extremal instantons satisfy a Bogomol’nyi bound. This motivates to add a specific boundary term to the original ‘purely scalar’ action, which ensures that its evaluation on instanton solutions give the same result as the dual ‘scalar-tensor’ action. We show that the instanton action obtained this way agrees with the ADM mass of the black hole obtained by lifting the solution to five dimensions. If instead we lift four-dimensional solutions to five dimensions without coupling to gravity, we again find that the mass of the resulting soliton is equal to the instanton action.

1.2 Overview

This paper is structured as follows. In Section 2.1 we introduce the class of Euclidean sigma models which we will use to generate solutions. The scalar target space is required to be para-Hermitian⁴ and to have n commuting shift isometries. In Section 2.2 we show that the Euclidean scalar equations of motion can be reduced to a set of linear harmonic equations

⁴The relevant concepts from para-complex geometry will be explained in Section 2.

by imposing the extremal instanton ansatz, which corresponds to restricting the scalar fields to vary along totally isotropic subspaces of the target space. The consistency of the solution leads to an integrability condition, which has two natural solutions. Either one restricts the solution to depend on one variable only. Since we require that solutions approach a vacuum at infinity, this implies spherical symmetry. While this does not impose conditions on the scalar metric, it excludes multi-centered solutions. The second, more interesting solution of the integrability condition requires that the scalar metric has a Hesse potential. In this case the target space is para-Kähler rather than only para-Hermitian. Since no constraint needs to be imposed on the solutions themselves, we obtain multi-centered solutions. In Section 2.3 we define instanton charges, which are the conserved charges corresponding to the n commuting shift symmetries, which are required in order to be able to lift the Euclidean sigma model to a five-dimensional gauge theory. Then we rederive the extremal instanton solutions from a different angle. By imposing that solutions carry finite instanton charge, we can ‘peel off’ one derivative from the field equations and reduce them to first order equations. As long as one does not impose spherical symmetry these are still (quasi-linear) partial differential equations. But once spherical symmetry is imposed, which we do in Section 2.4, the field equations reduce to first order gradient flow equations. We include some observations and remarks about the relation of our approach to the one based on first order rewritings, and to the Hamilton-Jacobi approach.

In Section 3 we discuss a dual version of Euclidean sigma models, where the n axionic scalars have been dualized into tensor fields. In the dual formulation the action of extremal instanton solutions is finite, positive and satisfies a Bogomol’nyi bound. To be precise, the finiteness of the action requires a suitable behaviour of the scalar fields at the centers of the harmonic functions. These conditions are further analyzed in Section 4. Instead of working with the dual ‘scalar-tensor’ action, one can add a boundary to the original ‘purely scalar’ action, which has the effect that one obtains the same finite non-vanishing instanton action for both actions.

In Section 4 we analyze two classes of Hesse potentials in more detail: homogeneous functions and logarithms of homogeneous functions. In these cases the asymptotic behaviour of the scalars at the centers and at infinity can be determined even if the field equations cannot be solved in closed form. The conditions which guarantee the finiteness of the instanton action are found explicitly for this class: the Hesse potential must be homogeneous of negative degree, or it must be the logarithm of a homogeneous function (of any degree). The instanton action can be expressed as a function of the instanton charges and of the asymptotic scalar fields, which has the standard form of a BPS mass formula. We can also find analogues of the stabilization equations and generalized stabilization equations known from BPS black holes. Solutions do not quite show fixed point behaviour, but the scalars run off to points at infinite affine parameter, with fixed finite ratios that are determined by the charges. We give various explicit examples of solutions, which include both rigidly and locally supersymmetric models as well as models which cannot have a supersymmetric extensions (the generic case).

In Section 5 we briefly discuss the lifting of four-dimensional Euclidean sigma models to five-dimensional field theories without gravity. The most interesting result is that the mass

of the resulting soliton equals the instanton action. Since instanton charges are electric charges from the five-dimensional point of view, the expression for the mass takes the same form as for the BPS mass in a supersymmetric theory. The special case of a cubic Hesse potential gives us the rigid para- r -map between the scalar geometries of five-dimensional vector multiplets and four-dimensional Euclidean vector multiplets. For general Hesse potentials we obtain a ‘para-version’ of the generalized rigid r -map which relates Hessian manifolds to para-Kähler manifolds with n commuting shift isometries.

In Section 6.1 we discuss the relation between four-dimensional Euclidean sigma models coupled to gravity, and five-dimensional Einstein-Maxwell type theories. We start in five dimensions, and present a generalized version of the very special real geometry of vector multiplets where the prepotential is allowed to be homogeneous of arbitrary degree. This is used to write down a class of Einstein-Maxwell type Lagrangians, which reduce over time to para-Kähler sigma models with n commuting shift isometries, coupled to gravity. This provides a generalized version of the local para- r -map, which includes the para- r -map between supersymmetric theories as a special case. We also indicate how the reduction over space results in a generalized version of the local r -map.

We then set up an instanton – black hole dictionary. Lifting extremal instanton solutions gives extremal black holes, and the ADM mass is shown to be equal to the instanton action in Section 6.2. In Section 6.3 we turn to the entropy of the black holes, which is non-vanishing or zero, depending on how many charges are switched on. The black hole entropy can be interpreted in the instanton picture by using a specific conformal frame for the four-dimensional metric, which is different from the Einstein frame. We call this frame the Kaluza-Klein frame, because it corresponds to a fixed time slice of the five-dimensional metric. In this frame the four-dimensional metric of extremal instantons is not flat, but only conformally flat, and the geometry can be interpreted as a semi-infinite wormhole. The Bekenstein-Hawking entropy of the black hole corresponds to the asymptotic size of the throat of the wormhole, and the degenerate case of black holes with vanishing entropy corresponds to wormholes with vanishing asymptotic size of the throat.

In Section 6.4 we illustrate the relation between extremal instantons and extremal black holes with several explicit examples. Then we show in full generality that the instanton attractor equations lift to black hole attractor equations, which have the same form as the stabilization equations and generalized stabilization equations of five-dimensional vector multiplets. In particular, we show that the ‘fixed-ratio run-away’ behaviour of four-dimensional scalar is equivalent to the proper fixed point behaviour of five-dimensional scalars.

In Appendix A we expand on the observation that target space geometries, which are obtained from a higher-dimensional theory by dimensional reduction over space or time, respectively, can be viewed as different real sections of one underlying complex target space. We explain the notion of ‘complexifying (para-)complex numbers’ and indicate that complex-Riemannian geometry is the appropriate framework for relating target spaces occurring in dimensional reduction over space and time by analytical continuation.

2. Sigma models with para-Hermitian target spaces

2.1 Motivation and discussion of the Euclidean action

The starting point for all subsequent constructions are sigma models of the form

$$S[\sigma, b]_{(0,4)} = \int d^4x \frac{1}{2} N_{IJ}(\sigma) (\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J) . \quad (2.1)$$

Space-time is taken to be flat Euclidean space E with indices $m = 1, 2, 3, 4$. The target space M is $2n$ -dimensional with coordinates σ^I, b^I , where $I = 1, \dots, n$. The matrix $N_{IJ}(\sigma)$ is assumed to be real, positive definite and only depends on half of the scalar fields. Thus the metric of M has n commuting isometries which act as shifts on the axionic scalars b^I :

$$b^I \rightarrow b^I + C^I , \quad (2.2)$$

where C^I are constants. The relative minus sign between the kinetic terms of the scalars σ^I and the axionic scalars b^I implies that the metric $N_{IJ} \oplus (-N_{IJ})$ of M has split signature (n, n) . There are two related reasons for considering Euclidean sigma models with split signature target spaces:

1. One approach to the definition of Euclidean actions combines the standard Wick rotation with an analytic continuation $b^I \rightarrow ib^I$ for axionic scalars [40]. D-instantons and other instanton solutions of string theory and supergravity are obtained as classical solutions of Euclidean actions of this type [41, 42, 43, 44, 45, 25].
2. Solitons, i.e. stationary, regular, finite energy solutions of $(n+1)$ -dimensional theories can be dimensionally reduced over time, resulting in instantons, i.e. regular finite action solutions of the reduced n -dimensional Euclidean theory. Conversely, one approach to the construction of solitons is to reduce the theory under consideration over time to obtain a simpler Euclidean theory, preferably a scalar sigma model. Instanton solutions of the Euclidean theory can then be lifted to solitons of the original theory. String theory has a large variety of solitonic solutions, which play a central role in establishing string dualities and thus obtaining information about the non-perturbative completion of the theory. Dimensional reduction has been used as a solution generating technique for some time in Einstein-Maxwell theory [30], supergravity [31] and string theory [49]. More recent applications include [33, 34, 24, 38]. We refer to [1, 32, 38] for a review of this method.

If we do not couple the sigma model (2.1) to gravity, then its lift to $1 + 4$ dimensions is⁵

$$S[\sigma, A, \dots]_{(1,4)} = \int d^5x \left(-\frac{1}{2} N_{IJ}(\sigma) \partial_\mu \sigma^I \partial^\mu \sigma^J - \frac{1}{4} N_{IJ}(\sigma) F_{\mu\nu}^I F^{J|\mu\nu} + \dots \right) . \quad (2.3)$$

⁵Dimensional lifting in the presence of gravity will be discussed later. As we will see the results obtained without coupling to gravity remain valid, provided that suitable restrictions on the scalar metric are imposed.

Here space-time is five-dimensional Minkowski space with indices $\mu, \nu, \dots = 0, 1, 2, 3, 4$ and $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I$ are abelian field strength. It is easy to see that (2.3) reduces to (-1) times the action (2.1) upon setting

$$\partial_0 \sigma^I = 0, \quad \partial_0 A_m^I = 0, \quad F_{mn}^I = 0,$$

identifying $b^I = A_0^I$ and dropping the integration over time. This type of reduction corresponds to the restriction to static and purely electric five-dimensional backgrounds. As indicated by the ‘dots’ in (2.3), the five-dimensional theory could have further terms as long as they do not contribute to static, purely electric field configurations involving the scalars and gauge fields. For example, the action for five-dimensional vector multiplets [36] contains a Chern-Simons term and fermionic terms, but these do not contribute to backgrounds which are static and where only scalars and electric field strength are excited. Note that there is a conventional minus sign between (2.3) and (2.1). Our conventions for Lorentzian actions are that the space-time metric is of the ‘mostly plus’ type, and that kinetic terms are positive definite. The convention for Euclidean actions is that the terms for the scalars σ^I are positive definite, while scalar fields obtained by temporal reduction of gauge fields have a negative definite action.

We can also reduce the five-dimensional action (2.3) over a space-like direction, leading to the following sigma model on four-dimensional Minkowski space:

$$S[\sigma, b]_{(1,3)} = - \int d^4x \frac{1}{2} N_{IJ}(\sigma) (\partial_{\bar{m}} \sigma^I \partial^{\bar{m}} \sigma^J + \partial_{\bar{m}} b^I \partial^{\bar{m}} b^J), \quad (2.4)$$

where $\bar{m} = 0, 1, 2, 3$. Note that we have discarded all terms in (2.3) which do not contribute to the scalar sigma model. By a Wick rotation we obtain the Euclidean action

$$S[\sigma, b]_{(0,4)}' = \int d^4x \frac{1}{2} N_{IJ}(\sigma) (\partial_m \sigma^I \partial^m \sigma^J + \partial_m b^I \partial^m b^J), \quad (2.5)$$

which is positive definite. Comparison to (2.1) shows explicitly that dimensional reduction over space followed by a Wick rotation is different from dimensional reduction over time. However, the two actions (2.1) and (2.5) are related by the analytic continuation $b^I \rightarrow ib^I$. In other words, two actions obtained by space-like and by time-like reduction respectively are related by a modified Wick rotation which acts non-trivially on axionic scalars.

For later reference, let us introduce the following notation for the target spaces of the actions we have encountered so far. The five-dimensional action (2.3) has an n -dimensional target space M_r with positive definite metric N_{IJ} . The four-dimensional actions (2.4) and (2.5) have a $2n$ -dimensional target space M' with positive definite metric $N_{IJ} \oplus N_{IJ}$, while (2.1) has a $2n$ -dimensional target space M with split signature metric $N_{IJ} \oplus (-N_{IJ})$.

The manifolds M and M' carry additional structures. For M' we can define complex coordinates

$$Y^I = \sigma^I + ib^I,$$

and we see that the target space M' is Hermitean:

$$S[Y]_{(0,4)}' = \int d^4x \frac{1}{2} N_{IJ}(Y + \bar{Y}) \partial_m Y^I \partial^m \bar{Y}^J. \quad (2.6)$$

This begs the question whether there is a similar additional structure for the indefinite target space M . And indeed, here one can define para-complex coordinates by

$$X^I = \sigma^I + e b^I ,$$

where the para-complex unit has the properties

$$e^2 = 1 , \quad \bar{e} = -e .$$

The theory of para-complex manifolds runs to a large extent parallel to the theory of complex manifolds. In particular, the concepts of para-Hermitian, para-Kähler, and special para-Kähler manifolds are analogous to their complex counterparts. We refer to [36, 37, 38] for a detailed account. Using para-complex coordinates, one sees that the action (2.1) has a para-Hermitian target space:

$$S[X]_{(0,4)} = \int d^4x \frac{1}{2} N_{IJ} (X + \bar{X}) \partial_m X^I \partial^m \bar{X}^J . \quad (2.7)$$

Thus actions of the type (2.1) have a target space which is para-Hermitian and has n commuting isometries acting as shifts. The latter implies that M can be obtained from an n -dimensional manifold M_r with positive definite metric, by applying temporal dimensional reduction to the corresponding action.

The two real actions (2.6) and (2.7) can be viewed as two different real forms of one underlying complex action. This is further explained in Appendix A. Complex actions are useful to get a more unified picture actions and solutions which are related by analytic continuation. In [51] complex actions for the ten-dimensional and eleven-dimensional maximal supergravity theories have been used to give a unified description of domain wall and cosmological solutions. There seems to be a close relation to the concept of fake supersymmetry. Complex actions seem also to be useful in understanding the Euclidean action of four-dimensional supergravity theories [38].

2.2 From harmonic maps to harmonic functions

Solving the equations of motion for a sigma model is equivalent to constructing a harmonic map from the space-time X to the scalar target space M . In general, both X and M can be (pseudo-)Riemannian manifolds. We restrict ourselves to the case where X is Euclidean space E equipped with its standard flat metric. Then the action of a general sigma model takes the form

$$S[\Phi]_{(0,4)} = \int d^4x N_{ij}(\Phi) \partial_m \Phi^i \partial^m \Phi^j ,$$

and the equations of motion can be brought to the form

$$\Delta \Phi^i + \Gamma_{jk}^i \partial_m \Phi^j \partial^m \Phi^k = 0 , \quad (2.8)$$

where Γ_{jk}^i are the Christoffel symbols of the metric N_{ij} of M . This is the coordinate form of the equation of a harmonic map $\Phi : E \rightarrow M$ from Euclidean space E to the (pseudo-)Riemannian target M .

One strategy for constructing such maps⁶ is to identify totally geodesic submanifolds $N \subset M$. A submanifold $N \subset M$ is called completely geodesic if every geodesics of N is also a geodesic of M . Then the embedding of N into M is a totally geodesic map, and since the composition of a harmonic map $E \rightarrow N$ with a totally geodesic map $N \rightarrow M$ is harmonic, it suffices to find harmonic maps $\phi : E \rightarrow N \subset M$ in order to solve the scalar equations of motion. We are interested in a criterion which guarantees that the solution of the harmonic map equation (2.8) can be expressed in terms of harmonic functions. This will happen in particular if the submanifold N is flat, so that the Christoffel symbols vanish identically if we use affine coordinates. Then we can parametrize the scalar fields such that the independent scalars ϕ^a , $a = 1, \dots, \dim N$ corresponds to affine coordinates on N , and the harmonic map equation reduces to

$$\Delta \phi^a = 0 . \quad (2.9)$$

If N has $\dim N < \dim M$, then the solution for the remaining $\dim M - \dim N$ scalar fields can be expressed in terms of the solution for the ϕ^a . The dimension of N controls the number of independent harmonic functions which occur in the solution.

We will now investigate under which conditions the reduction of the equations of motion to decoupled harmonic equations can be achieved, assuming that the target manifold M is para-Hermitian and has n commuting shift symmetries. In this case it is convenient to write the equations of motion in terms of the real fields σ^I , b^I . By variation of the action (2.1) we obtain:

$$\begin{aligned} \partial^m (N_{IJ} \partial_m \sigma^J) - \frac{1}{2} \partial_I N_{JK} (\partial_m \sigma^J \partial^m \sigma^K - \partial_m b^J \partial^m b^K) &= 0 , \\ \partial^m (N_{IJ} \partial_m b^J) &= 0 . \end{aligned} \quad (2.10)$$

This could be cast into the form (2.8), but in the present form it is manifest that a drastic simplification occurs if we impose that

$$\partial_m \sigma^I = \pm \partial_m b^I . \quad (2.11)$$

In this case the two equations (2.10) collapse into

$$\partial^m (N_{IJ} \partial_m \sigma^J) = 0 , \quad (2.12)$$

which is very close to the harmonic equation. We will refer to the condition (2.11) as the extremal instanton ansatz. Geometrically, the extremal instanton ansatz implies that the scalar fields are restricted to vary along the null directions of the metric of M . In other words, the scalars take values in a submanifold $N \subset M$ which is completely isotropic.

The extremal instanton ansatz has the consequence that the energy momentum tensor vanishes identically. The ‘improved’, symmetric energy momentum tensor for the action (2.1) is obtained by variation with respect to a Riemannian background metric on E :

$$T_{mn} = N_{IJ} (\partial_m \sigma^I \partial_n \sigma^J - \partial_m b^I \partial_n b^J) - \frac{1}{2} \delta_{mn} N_{IJ} (\partial_l \sigma^I \partial^l \sigma^J - \partial_l b^I \partial^l b^J) . \quad (2.13)$$

⁶See [1, 32, 38] for a more detailed review.

Since we are in four dimensions (more generally in > 2 dimensions), the vanishing of T_{mn} is equivalent to

$$N_{IJ} (\partial_m \sigma^I \partial_n \sigma^J - \partial_m b^I \partial_n b^J) = 0, \quad (2.14)$$

which means that the scalar fields vary along the null directions of $N_{IJ} \oplus (-N_{IJ})$. More precisely, depending on the choice of sign in (2.11), the scalar fields vary along the eigendirection of the para-complex structure with eigenvalue $(+1)$ or (-1) , respectively. Thus the submanifold N is the integral manifold of an eigendistribution of the para-complex structure.

The vanishing of the energy momentum tensor has the important consequence that solutions of (2.1) which satisfy (2.11) remain solutions, without any modification, if we couple the sigma model to gravity,

$$S[g, \sigma, b]_{(0,4)} = \int d^4x \sqrt{g} \frac{1}{2} (-R + N_{IJ} \partial_m \sigma^I \partial^m \sigma^J - N_{IJ} \partial_m b^I \partial^m b^J). \quad (2.15)$$

Since the energy momentum tensor vanishes, it is consistent to solve the Einstein equation by taking the metric to be flat, $g_{mn} = \delta_{mn}$. Thus the instanton solutions we find are solutions of sigma models coupled to gravity (2.15), subject to the ‘Hamiltonian constraint’ $T_{mn} = 0$. As we will discuss in more detail later, instanton solutions of (2.1) can therefore be lifted consistently to solutions of five-dimensional gravity coupled to matter. In this case the lifting works somewhat differently than in the rigid case, because one has to take into account the Kaluza-Klein scalar. The resulting five-dimensional solutions are not flat, but have a conformally flat four-dimensional part, as is typical for BPS solutions. One particular class of lifted solutions are extremal static five-dimensional black holes. The theories and solutions obtainable from (2.1) include all five-dimensional supergravity theories with abelian vector multiplets and their BPS black hole solutions. We will refer to instanton solutions obtained by the extremal instanton ansatz as extremal. One reason for this choice of terminology is that they can be lifted to extremal black hole solutions. Another reason is the saturation of a Bogomol’nyi bound for the action, which will be discussed in Section 3.

The indefiniteness of the metric of M is essential for obtaining non-trivial solutions of the scalar equations of motion with vanishing energy momentum tensor. For a positive (or negative) definite scalar target space metric, $T_{mn} = 0$ would imply that all scalar fields have to be constant.

The extremal instanton ansatz (2.11) is sufficient but not necessary for the vanishing of the energy momentum tensor and the reduction of the equations of motion to (2.12). If the metric N_{IJ} is invariant under transformations of the form

$$N_{IJ} \rightarrow N_{KL} R^K{}_I R^L{}_J, \quad (2.16)$$

where $R^I{}_J$ is a constant matrix, already the generalized instanton ansatz⁷ (2.11)

$$\sigma^I = R^I{}_J b^J \quad (2.17)$$

⁷‘Generalized extremal instanton ansatz’ would be more accurate, but we will use ‘generalized instanton ansatz’ for convenience.

implies $T_{mn} = 0$ and (2.12). Geometrically, the transformation (2.16) corresponds to an isometry of $N_{IJ} \oplus (-N_{IJ})$ which acts by

$$\sigma^I \rightarrow \sigma^I, \quad b^I \rightarrow R^I_J b^J.$$

The relation (2.17) has occurred previously in the context of extremal black hole solutions of supergravity, where $R^I_J \neq \delta^I_J$ corresponds to non-BPS solutions [22, 23]. The simplest examples of non-BPS black holes correspond to flipping some of the charges of the black hole, which corresponds to diagonal R -matrices with entries ± 1 . Geometrically, this means that some of the fields vary along the $(+1)$ -eigendirections of the para-complex structure while the rest varies along the (-1) -eigendirections. In this way the distinction between BPS and non-BPS extremal solutions in supergravity can be understood geometrically and extended to a larger class of non-supersymmetric theories.

Let us now investigate the reduced equations of motion (2.12), which remain to be solved after imposing the extremal instanton ansatz (2.11) or its generalization (2.17):

$$\partial^m (N_{IJ} \partial_m \sigma^J) = 0.$$

This reduces to a set of n harmonic equations, provided there exist ‘dual fields’ σ_I with the property

$$\partial_m \sigma_I = N_{IJ} \partial_m \sigma^J.$$

The existence of such dual fields implies the integrability condition

$$\partial_{[n} (N_{IJ} \partial_{m]} \sigma^J) = 0.$$

The same condition has been observed in the context of five-dimensional black hole solutions in [24]. Since $\partial_{[n} \partial_{m]} \sigma^J = 0$, the integrability condition is equivalent to

$$\partial_{[n} N_{IJ} \partial_{m]} \sigma^J = \partial_K N_{IJ} \partial_{[n} \sigma^K \partial_{m]} \sigma^J = 0. \quad (2.18)$$

There are two strategies for solving this constraint. The first is to restrict the solution $\sigma^I(x)$ while not making assumptions about the metric N_{IJ} . If we assume that the solution only depends on one of the coordinates of E , then (2.18) is solved automatically. The most natural assumption is spherical symmetry, $\sigma^I = \sigma^I(r)$, where r is a radial coordinate, as this admits solutions which asymptotically approach ground states $\sigma^I_{\text{vac}} = \text{const}$ at infinity. In this case the explicit solutions of $\Delta \sigma_I = 0$ are single-centered harmonic functions,

$$\sigma_I = H_I(r) = h_I + \frac{q_I}{r^2},$$

where h_I and q_I are constants. The constants h_I specify the values of σ_I at infinity. As we will see in the next section the parameters q_I are charges. Such solutions can be interpreted as describing an instanton with charges q_I located at $r = 0$.⁸ Geometrically, this type of solution corresponds to a situation where the scalars flow along a null geodesic curve in M .

⁸At $r = 0$ the fields σ_I take singular values, and the equations of motion (2.12) are not satisfied, unless explicit source terms are added. This is analogous to electric point charges in electrostatics.

The second strategy is to make no assumption about the solution. This is compulsory if we want that there are multi-centered solutions,

$$\sigma_I(x) = H_I(x) = h_I + \sum_{a=1}^N \frac{q_{aI}}{|x - x_a|^2} , \quad (2.19)$$

where h_I, q_{aI} are constants and where $x, x_a \in E$. Such solutions correspond to N instantons with charges q_{aI} , which are located at the positions x_a . For multi-centered solutions we cannot impose spherical symmetry but the integrability condition (2.18) can still be solved by imposing the condition

$$\partial_{[K} N_{I]J} = 0 \quad (2.20)$$

on the scalar metric. This is equivalent to requiring that the first derivatives $\partial_K N_{IJ}$ of the metric are completely symmetric, or, again equivalently, that the Christoffel symbols of the first kind $\Gamma_{IJ|K}$ are completely symmetric. Finally, by applying the Poincaré lemma twice, we see that (2.20) is locally equivalent to the existence of a Hesse potential $\mathcal{V}(\sigma)$:

$$N_{IJ} = \frac{\partial^2 \mathcal{V}}{\partial \sigma^I \partial \sigma^J} . \quad (2.21)$$

A coordinate-free formulation is obtained by observing that the local existence of a Hesse potential is equivalent to the existence of a flat, torsion-free connection ∇ which has the property that ∇g , where g is the metric, is a completely symmetric rank 3 tensor field. This is the definition of a Hessian metric given in [39]. They also observed that the affine special real manifolds which are the target spaces of rigid five-dimensional vector multiplets are special Hessian manifolds where the cubic form ∇g is parallel with respect to ∇ . It is easy to see why supersymmetry requires this additional condition. Supersymmetry implies the presence of a Chern-Simons term, whose coefficient is given by ∇g . Gauge invariance requires that this coefficient is covariantly constant. In affine coordinates, this becomes the well known condition that the third derivatives of the Hesse potential (which for rigid supersymmetry is identical with the prepotential) must be constant. Hence the Hesse potential must be a cubic polynomial. In this paper we consider more general Hesse potentials, but since the models are not supersymmetric, there is no fixed relation between the Chern Simons term (if any is present) and other terms in the Lagrangian. In the purely electric background that we consider, a Chern-Simons does not contribute, and therefore we do not need to investigate whether a Chern-Simons term could or should be added.

The dimensional reduction of models with general Hessian target spaces leads to a generalization of the rigid version of the r -map [39]. Recall that the r -map relates the target spaces of five-dimensional and four-dimensional vector multiplets [50]. The r -map can be derived by dimensionally reducing the vector multiplet action from five to four dimensions, and depending on whether one considers supersymmetric field theories or a supergravity theories one obtains a rigid (also called affine) or local (also called projective) version of the r -map. Affine (projective) very special real manifolds are mapped to affine (projective) special Kähler manifolds, respectively. The generalized rigid r -map of [39] is obtained by relaxing the constraint that the scalar target geometry of the five dimensional

theory is very special real and only requiring it to be Hessian. In the notation of our paper the resulting generalized r -map is obtained by reducing (2.3) to (2.5) while imposing that $N_{IJ}(\sigma)$ satisfies (2.21). As shown in [39] the resulting target space M' of the four-dimensional theory is a Kähler manifold with n commuting shift isometries. We already noted that M' is Hermitean. To check that it is Kähler we go to complex coordinates $Y^I = \sigma^I + ib^I$ and verify by explicit calculation that

$$K(Y, \bar{Y}) = K(Y + \bar{Y}) = 4\mathcal{V}(\sigma(Y, \bar{Y}))$$

is a Kähler potential for the metric $N_{IJ} \oplus N_{IJ}$ of M' . Note that [39] prove that the relation between Hessian manifolds (M_r, N_{IJ}) and Kähler manifolds $(M', N_{IJ} \oplus N_{IJ})$ is one-to-one: any Kähler manifold with n commuting shift isometries can be obtained from a Hessian manifold by the generalized r -map.

By reducing (2.3), with Hessian N_{IJ} , over time rather than space, we obtain (2.1) and a para-version (or temporal version) of the generalized rigid r -map. As shown in [36], the rigid para- r -map relates affine very special real manifolds to affine special para-Kähler manifolds with a cubic prepotential. If we only impose that M_r is Hessian, then M is a para-Kähler manifold with n commuting shift isometries. We have already seen that metric $N_{IJ} \oplus (-N_{IJ})$ is para-Hermitean. To see that it is para-Kähler we go to para-complex coordinates $X^I = \sigma^I + eb^I$ and verify that

$$K(X, \bar{X}) = K(X + \bar{X}) = 4\mathcal{V}(\sigma(X, \bar{X}))$$

is a para-Kähler potential. We expect that every para-Kähler metric with n commuting shift isometries can be obtained in this way.

To conclude this section, let us discuss how our class of solutions fits into the general set up of constructing harmonic maps $E \rightarrow M$ by finding totally geodesic embeddings $N \subset M$ and harmonic maps $E \rightarrow N$. The extremal instanton ansatz implies that in our construction N is totally isotropic. Although the induced metric of N is totally degenerate one can still construct harmonic maps $E \rightarrow M$, by decomposing ‘harmonic maps’⁹ $E \rightarrow N$ with totally geodesic embeddings $N \subset M$ [38]. Our explicit calculation using the coordinates σ^I, b^I shows that the totally isotropic submanifolds defined by the extremal instanton ansatz must be totally geodesic. This can also be understood as follows. The submanifolds defined by the ansatz (2.11) are eigendistributions of the para-complex structure of M . The integrability condition (2.20) implies that M is para-Kähler, and therefore the eigendistributions are integrable and parallel with respect to the Levi-Civita connection. Such submanifolds are in particular totally geodesic.

By explicit calculation we have also seen that N must be flat. This can again be understood geometrically. In complete analogy to Kähler manifolds, the Riemann tensor of a para-Kähler manifold has a particular index structure, when written in para-complex coordinates. The only non-vanishing components are those where both pairs of indices are

⁹Strictly speaking, we should not call this a harmonic map if N is totally isotropic, because the definition of a harmonic map requires that source and target manifolds are equipped with non-degenerate metrics. However, the relevant point is that the composed map $E \rightarrow N \subset M$ is harmonic.

of mixed type, i.e. one para-holomorphic and one anti-para-holomorphic index. However, the pullback of the Riemann tensor to an eigendistribution of the para-complex structure is of pure type, i.e. the non-vanishing components have either only para-holomorphic or only anti-para-holomorphic indices. Therefore the pull back of the Riemann tensor onto these eigendistributions vanishes. Thus the pulled back connection is flat, and the harmonic map equations must reduce to harmonic equations when expressed in affine coordinates. We can also understand why the existence of single-centered solutions does not impose constraints on the scalar target metric. In this case N is a null geodesic curve, and therefore it is flat for any choice of the metric of M .

The additional feature which is required for the existence of multi-centered solutions is the existence of a potential. For many purposes it is convenient to use real coordinates σ^I, b^I and to work with the Hesse potential $\mathcal{V}(\sigma)$. The affine coordinates on N , in terms of which the equations of motion reduce to decoupled harmonic equations $\Delta\sigma_I = 0$ are given by the first derivatives of the Hesse potential

$$\sigma_I \simeq \frac{\partial \mathcal{V}}{\partial \sigma^I} . \quad (2.22)$$

This is clear because the application of the partial derivative ∂_m on (2.22) gives the integrability condition (2.18). In (2.22) we have left the constant proportionality undetermined, so that we can later fix it to convenient numerical values case by case. Given the Hesse potential we have an explicit formula for the dual fields σ_I in terms of the scalars σ^I . In general it is not possible to give explicit expressions for the scalars σ^I as functions of the dual scalars σ_I . Hence, while instanton solutions are completely determined by harmonic functions H_I through a set of algebraic relations, it is not possible in general to express the solution in terms of the H_I in closed form. However, we will see that this does not prevent us from understanding many features of the solutions. Moreover, if the Hesse potential is sufficiently simple, explicit expressions can be obtained. Examples will be given later.

2.3 From instanton charges to harmonic functions

We have already anticipated that the parameters q_I or q_{Ia} occurring in the harmonic functions H_I can be interpreted as charges. In this section we provide the definition of instanton charges and derive the instanton solutions from a slightly different perspective. It has been observed in the literature on extremal non-BPS black holes that if solutions can be expressed in terms of harmonic functions then the equations of motion can often be reduced from second to first order equations [22, 23, 24]. Our derivation will show that these two properties are related through the existence of conserved charges: imposing that solutions carry *finite* instanton charges implies that the equations of motion can be replaced by first order equations.

The symmetry of the target manifold M under constant shifts $b^I \rightarrow b^I + C^I$ implies the existence of n charges, which we call instanton charges. As we will see later these lift to five-dimensional electric charges. The current associated to the shift symmetry is

$$j_I = \partial_m (N_{IJ}(\sigma) \partial^m b^J) .$$

It is ‘conserved’ in the sense that the Hodge-dual four-form is closed. The charge obtained by integrating this current over four-dimensional Euclidean space is

$$Q_I = \int d^4x j_I . \quad (2.23)$$

Since j_I is a total derivative, the charge Q_I can be re-written as a surface charge, as is typical for gauge theories:

$$Q_I = \oint d^3\Sigma^m (N_{IJ}(\sigma) \partial_m b^J) .$$

The integral is performed over a topological three-sphere which encloses all sources. Note that explicit sources are needed to have non-vanishing instanton charges, because the equation of motion (2.10) for b^I implies $j_I = 0$. To obtain non-trivial solutions we allow the presence of pointlike (δ -function type) sources. Pointlike sources in Euclidean theories are referred to as (-1) -branes. The existence of solutions carrying (-1) -brane charge is taken as evidence that the theory should be extended by adding (-1) -branes. This is the philosophy underlying field theory and string dualities, for which our class of actions provides models.

Solutions with non-vanishing instanton charge must have a particular asymptotic behaviour. We assume that the sources are contained in a finite region and take the limit $r \rightarrow \infty$, where r is a radial coordinate with origin within this region. The integrand can be expanded in powers of $\frac{1}{r}$, and we assume that the contribution of the leading term to the charges Q_I is non-vanishing and finite.¹⁰ This implies that subleading terms in $\frac{1}{r}$ do not contribute to the charge. Since the leading term is independent of the angles, we take the integration surface to be a three-sphere S_r^3 of radius r and integrate over the angles. The resulting charge is

$$Q_I = \text{vol}(S_1^3) \lim_{r \rightarrow \infty} (r^3 N_{IJ}(\sigma) \partial_r b^J) , \quad (2.24)$$

where $\text{vol}(S_1^3) = 2\pi^2$ is the volume of the unit three-sphere. Since we assume that Q_I is neither infinite nor zero, it follows that the integrand $N_{IJ} \partial_r b^J$ falls off like $\frac{1}{r^3}$:

$$N_{IJ} \partial_r b^J = \frac{1}{2\pi^2} \frac{Q_I}{r^3} + \dots ,$$

where the omitted terms are of order $\frac{1}{r^4}$. Now we observe that the leading term in $N_{IJ} \partial_r b^J$ is the derivative of a spherically symmetric harmonic function $\tilde{H}_I(r)$:

$$N_{IJ} \partial_r b^J = \frac{1}{2\pi^2} \frac{Q_I}{r^3} + \dots = \partial_r \tilde{H}_I(r) + \dots , \quad (2.25)$$

with

$$\tilde{H}_I(r) = \frac{\tilde{q}_I}{r^2} + \tilde{h}_I ,$$

¹⁰The expansion in $\frac{1}{r}$ is of course a version of the multipole expansion. In fact, from the five-dimensional point of view it is literally the multipole expansion of a discrete charge density contained in a finite region.

where \tilde{h}_I are constants and where $\tilde{q}_I = -\frac{1}{4\pi^2}Q_I$ are proportional to the charges. For simplicity we will refer to the parameters \tilde{q}_I as charges.

While the leading term of the expanded solution is automatically spherically symmetric, the full solution may not be. This leads to the distinction of two cases, precisely as in the previous section. If we impose that the full solution is spherically symmetric, then we obtain a solution to the equations of motion by setting all subleading terms to zero, and imposing the extremal instanton ansatz (2.11) or its generalized version (2.17):

$$N_{IJ}\partial_r b^J = \partial_r \tilde{H}_I(r) .$$

This solution is spherically symmetric and at $r = 0$ the equations of motion need to be modified by a δ -function type source term with coefficient q_I . The source is interpreted as a (-1) -brane of total charge \tilde{q}_I , which is located at the center $r = 0$ of the harmonic function.

If we do not assume that the full solution is spherically symmetric, then we need to find solutions of (2.12) with asymptotics (2.25) subject to the (generalized) extremal instanton ansatz. Such solutions are obtained if

$$N_{IJ}\partial_m b^J = \partial_m \tilde{H}_I(x) , \quad (2.26)$$

where $\tilde{H}_I(x)$ are harmonic functions. Since the right hand side is a total derivative, we need to impose an integrability condition equivalent to (2.20), and thus we recover the condition that the scalar metric $N_{IJ} \oplus (-N_{IJ})$ of M must be a para-Kähler metric. Assuming this, we have managed to reduce the second order equations of motion (2.12) to the first order quasilinear partial differential equations

$$\partial_m b^I = N^{IJ}\partial_m \tilde{H}_J(x) , \quad (2.27)$$

where N^{IJ} is the inverse of N_{IJ} . From our derivation it is clear that the crucial ingredient for the reduction of the order of the equation of motion is the existence of the charges Q_I , which can be used to prescribe the asymptotic behaviour of the solution, and ‘to peel off’ one derivative from the equation of motion, provided that the integrability condition (2.18) holds.

Solutions with the correct asymptotics are given by multi-centered harmonic functions

$$\tilde{H}_I(x) = \tilde{h}_I + \sum_{a=1}^N \frac{\tilde{q}_{aI}}{|x - x_a|^2} \quad (2.28)$$

where $x, x_a \in \mathbb{R}^4$. They correspond to N (-1) -branes with charges \tilde{q}_{aI} , which located at the centers x_a . For $|x| \rightarrow \infty$ the leading term is

$$\tilde{H}_I(x) \approx \frac{1}{|x|^2} \sum_{a=1}^N \tilde{q}_{aI} + \mathcal{O}(|x|^{-3}) . \quad (2.29)$$

Thus the total instanton charges of such a configuration are $\tilde{q}_I = \sum_{a=1}^N \tilde{q}_{aI}$.

The relation between this version of the solution and the one given in the previous section is provided by the (generalized) extremal instanton ansatz. Observe that

$$\partial_r \sigma_I = N_{IJ} \partial_r \sigma^J = N_{IJ} R_K^J \partial_r b^K = R_I^J N_{JK} \partial_r b^K ,$$

where R_I^J is the inverse of the transposed of R^I_J . Comparing the solutions (2.19) and (2.26) we conclude

$$\partial_m H_I = R_I^J \partial_m \tilde{H}_J ,$$

which implies $H_I = R_I^J \tilde{H}_J$ up to an additive constant. This constant reflects the shift symmetry $b^I \rightarrow b^I + C^I$. However the coefficients of the non-constant terms in the harmonic functions are not ambiguous and related by the rotation matrix R_I^J . In particular the instanton charges q_I and \tilde{q}_I are related by

$$q_I = R_I^J \tilde{q}_J .$$

Thus we have seen that the reduction of the equations of motion to the decoupled harmonic equations and the reduction of the equations of motion from second to first order differential equations result from the same integrability condition (2.18). The integrability condition can be solved by either imposing that the solution is spherically symmetric, or by restricting the target geometry to be para-Kähler.

2.4 Spherically symmetric solutions and flow equations

In this section we take a closer look at spherically symmetric solutions. For spherically symmetric black hole solutions (and other related solitonic solutions), the reduction of the equation of motion to first order equations was first noticed for BPS solutions. In this context the first order equations are known as (generalized) attractor equations, or (gradient) flow equations. Later it was realized that a first order rewriting is often also possible for non-BPS solutions, and leads to first order flow equations which are driven by potential, which generalizes the $N = 2$ central charge [22, 23, 24]. Let us therefore explain how gradient flow equations fit into our framework.

We have seen previously that if we impose that solutions carry non-vanishing instanton charge (and satisfy the integrability condition (2.18) which is trivial for spherically symmetric solutions), then the second order equations of motion reduce to the first order quasilinear partial differential equations (2.27). If we impose spherical symmetry the equations reduce further to the first order quasilinear ordinary differential equations

$$\sigma^{I'} = N^{IJ}(\sigma) H'_J(r) = N^{IJ}(\sigma) \frac{d}{dr} \left(\frac{q_J}{r^2} + h_J \right) ,$$

where $f' = \frac{df}{dr}$. The standard form of the flow equations is obtained by introducing the new coordinate $\tau = \frac{1}{r^2}$:

$$\dot{\sigma}^I = N^{IJ}(\sigma) \frac{d}{d\tau} (q_J \tau + h_J) = N^{IJ} q_J ,$$

where $\dot{f} = \frac{df}{d\tau}$. By introducing the function

$$W = q_J \sigma^J$$

and obtains the gradient flow equations

$$\dot{\sigma}^I = N^{IJ} \partial_J W . \quad (2.30)$$

In terms of the instanton charges $Q_I \propto \tilde{q}_I$, the ‘superpotential’ is

$$W = R_I{}^J \tilde{q}_J \sigma^I .$$

This form is familiar from black holes [22, 23, 24]. If the underlying theory is supersymmetric, and if the R -matrix is proportional to the identity, then

$$W = \pm Z = \pm \tilde{q}_I \sigma^I ,$$

where Z is the real central charge of the supersymmetry algebra of the underlying five-dimensional theory. Z is also one of the two real supercharges of the four-dimensional Euclidean supersymmetric theory obtained by reduction over time.

The new coordinate τ has a simple geometrical interpretation. To see this consider the version of the equations of motion which involve the dual scalars σ_I :

$$\Delta \sigma_I = 0 .$$

This is the harmonic equation for a map from space-time E into a flat submanifold $N \subset M$, written in terms of affine coordinates on N . For spherically symmetric solutions, this takes the form

$$\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} \right) \sigma_I = 0 .$$

This is the geodesic equation for a curve on a flat submanifold $N \subset M$. The presence of the second term shows that r is not an affine curve parameter. However, one can always introduce an affine parameter τ , which is unique up to affine transformations, such that equations reduces to

$$\frac{\partial^2}{\partial \tau^2} \sigma_I = 0 .$$

It is easy to see that for the case at hand the affine parameters are

$$\tau = \frac{a}{r^2} + b ,$$

where $a \neq 0$ and b are constants. Thus $r \rightarrow \tau = \frac{1}{r^2}$ is a reparametrization of the geodesic which brings it to affine form. The solution takes the particularly simple form of harmonic functions in one variable,

$$\sigma_I(\tau) = q_I \tau + \sigma_I(0) .$$

For completeness, let us review an alternative derivation of the flow equations, which uses a variant of the Bogomol’nyi trick and which is used frequently in the literature on

non-BPS extremal black holes (see for example [22, 23, 24]). In spherically symmetric backgrounds $ds^2 = dr^2 + r^2 d\Omega_{(3)}^2$ the action (2.1) can be reduced to the one-dimensional action

$$S[\sigma, b]_{(0,1)} = \int r^3 dr \frac{1}{2} N_{IJ} (\sigma^{I'} \sigma^{J'} - b^{I'} b^{J'}) .$$

Then one tries to rewrite this action as an (alternating) sum of perfect squares plus boundary terms. The factor r^3 can be eliminated by going to the affine curve parameter $\tau = \frac{1}{r^2}$:

$$S[\sigma, b]_{(0,1)} = \frac{1}{2} \int d\tau N_{IJ} (\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J) . \quad (2.31)$$

The Euler-Lagrange equations of this action need to be supplemented by the constraint

$$\mathcal{H} = \frac{1}{2} N_{IJ} (\dot{\sigma}^I \dot{\sigma}^J - \dot{b}^I \dot{b}^J) = 0 , \quad (2.32)$$

which implies that the solution is extremal.¹¹ From the five-dimensional point of view, this constraint imposes the Einstein equations, and therefore it is analogous to the four-dimensional constraint $T_{mn} = 0$. Note that \mathcal{H} in (2.32) is the Hamiltonian of the one-dimensional action. The canonical momenta

$$p_I = \frac{\partial \mathcal{L}}{\partial \dot{\sigma}^I} = N_{IJ} \dot{\sigma}^J \quad \text{and} \quad \tilde{p}_I = \frac{\partial \mathcal{L}}{\partial \dot{b}^I} = N_{IJ} \dot{b}^J \quad (2.33)$$

are conserved and agree with the charges: $p_I = q_I$ and $\tilde{p}_I = \tilde{q}_I$. Since the Lagrangian is quadratic in the velocities and does not contain a potential, the Hamiltonian coincides with the Lagrangian.

The first order form of the equations of motion can be obtained by rewriting the Lagrangian as an (alternating) sum of squares, up to boundary terms [22, 23, 24]:

$$\begin{aligned} S[\sigma, b]_{(0,1)} = \frac{1}{2} \int d\tau [& N_{IJ} (\dot{\sigma}^I - N^{IK} q_K) (\dot{\sigma}^J - N^{JL} q_L) \\ & - N_{IJ} (\dot{b}^I - N^{IK} \tilde{q}_K) (\dot{b}^J - N^{JL} \tilde{q}_L)] + \text{boundary terms} , \end{aligned} \quad (2.34)$$

where the constants q_I and \tilde{q}_I are related by $q_I = R_I^J \tilde{q}_J$. Since the boundary terms do not contribute to the equations of motion, a subclass of solutions is obtained by setting both squares to zero. This is equivalent to the combined flow equations for σ^I and b^I , or to the generalized instanton ansatz $\dot{\sigma}^I = R^I_J \dot{b}^J$ together with the flow equations for the independent scalars.

Reduced scalar manifold, geodesic potential, and remarks on the Hamilton-Jacobi formalism

So far we have worked on the scalar manifold M , which is parametrized by the $2n$ scalars σ^I and b^I . One approach used frequently in the literature is to eliminate the b^I by their equations of motion, which results in an effective potential for the σ^I which contains the

¹¹In general the constraint is $H = c^2$, where c is a constant, but the case $c \neq 0$ corresponds to non-extremal solutions [14], which we do not consider in this paper.

charges as parameters [14]. The resulting equation for σ describes geodesic motion with a non-trivial potential on the n -dimensional manifold M_r . We briefly review this approach in order to explain how our work relates to [27], who applied the Hamilton-Jacobi formalism to spherically symmetric, static black holes.¹² To facilitate the comparison, it is convenient to write the Euler-Lagrange equations of the action (2.31) in the following form:

$$\ddot{\sigma}^I + \Gamma_{JK}^I \dot{\sigma}^J \dot{\sigma}^K + \frac{1}{2} N^{IL} \partial_L N_{JK} \dot{b}^J \dot{b}^K = 0 , \quad (2.35)$$

$$\frac{d}{d\tau} (N_{IJ} \dot{b}^J) = 0 .$$

Here Γ_{JK}^I are the Christoffel symbols of the metric $N_{IJ}(\sigma)$ on the manifold M_r . While the combined set of equations is the geodesic equation for the metric $N_{IJ} \oplus (-N_{IJ})$ on the manifold M , one can use the fact that N_{IJ} is independent of the b^I to eliminate the b^I and thus obtain a geodesic equation with potential on M_r . The equations of motion of the b^I state that the quantities

$$\tilde{q}_I = N_{IJ} \dot{b}^J$$

are conserved. In fact, the \tilde{q}_I are the conserved axionic charges introduced previously. Using this the equations (2.35) reduce to

$$\ddot{\sigma}^I + \Gamma_{JK}^I \dot{\sigma}^J \dot{\sigma}^K + \frac{1}{2} N^{IL} \partial_L N_{JK} N^{JM} N^{KN} \tilde{q}_M \tilde{q}_N = 0 . \quad (2.36)$$

The constraint (2.32) now takes the form

$$\mathcal{H} = \frac{1}{2} (N_{IJ} \dot{\sigma}^I \dot{\sigma}^J - N^{IJ} \tilde{q}_I \tilde{q}_J) = 0 . \quad (2.37)$$

Expressing this in terms of the canonical momenta $p_I = N_{IJ} \dot{\sigma}^J$ and defining the ‘geodesic potential’

$$V(\sigma)_{\tilde{q}} = N^{IJ} \tilde{q}_I \tilde{q}_J , \quad (2.38)$$

the Hamiltonian constraint becomes

$$\tilde{\mathcal{H}}(\sigma, p) = \frac{1}{2} (p_I N^{IJ} p_J - V(\sigma)_{\tilde{q}}) = 0 .$$

The geodesic potential is positive definite for positive definite N_{IJ} [14, 27]. The relative minus sign between the ‘kinetic’ term and the potential is due to the fact that our ‘time’ is actually a space-like, radial coordinate. The associated action and Lagrangian are given by

$$\tilde{S}[\sigma]_{(0,1)} = \frac{1}{2} \int d\tau (N_{IJ} \dot{\sigma}^I \dot{\sigma}^J + V(\sigma)_{\tilde{q}}) . \quad (2.39)$$

Note that this action is *not* obtained by substituting the definition of the geodesic potential into (2.31), which would lead to a different sign in front of the potential. Rather, the two Hamiltonians are related through eliminating the b^I by their equations of motion, and the associated Lagrangians are in turn given as Legendre transforms. This distinction is

¹²While [27] also consider non-extremal black holes, we restrict ourselves to the extremal case.

crucial, since the elimination of the b^I leads to a non-trivial potential. To check that the procedure is correct observe that the Euler-Lagrange equations of (2.39)

$$\ddot{\sigma}^I + \Gamma_{JK}^I \dot{\sigma}^J \dot{\sigma}^K - \frac{1}{2} N^{IL} \partial_L V(\sigma)_{\tilde{q}} = 0$$

agree with (2.36), which were obtained from the Euler-Lagrange equations (2.35) of the action (2.31) by eliminating the b^I through their equation of motion.

The problem investigated in [27] is the following: given an action of the form (2.39), how can one find new coordinates Σ^I and momenta P_I , such that the new momenta are conserved? By Hamilton-Jacobi theory this can be achieved by finding a suitable generating function $\tilde{W}(\sigma, P, \tau)$ of the old coordinates and new momenta. This function must in particular satisfy $p_I = \frac{\partial \tilde{W}}{\partial \sigma^I}$ and $\Sigma^I = \frac{\partial \tilde{W}}{\partial P_I}$. Since $p_I = N_{IJ} \dot{\sigma}^J$ this leads to a first order gradient flow driven by the generating \tilde{W} : $\dot{\sigma}^I = N^{IJ} \partial_J \tilde{W}$ [27]. For extremal black holes the generating function is independent of ‘time’ τ [27].

As we have seen above, the coordinates σ^I which we use throughout this paper have associated canonical momenta p_I which are proportional to the charges and hence conserved. This is due to the extremal instanton ansatz, which solves the constraint $\mathcal{H} = 0$ by imposing that $\dot{\sigma}^I$ and \dot{b}^I are proportional up to the constant matrix $R^I{}_J$. Since the momenta associated with the b^I are conserved as a consequence of the shift symmetries, the extremal instanton ansatz implies that the p_I are conserved as well. Above we derived gradient flow equations (2.30) which are driven by the ‘superpotential’ $W = q_I \sigma^I$. As is easily verified, we can interpret this function as the generating function $\tilde{W} = P_I \sigma^I$ of the trivial canonical transformation $\Sigma^I = \sigma^I$, $P_I = p_I$. The triviality of the Hamiltonian-Jacobi problem reflects that we are already working, for extremal black holes, in the coordinate system adapted to the symmetries. Note that this does not require any assumption on the geometry of M_r , because for spherically symmetric black holes the integrability condition does not impose constraints on the scalar geometry.

In the case where the manifold M_r is Hessian, we can go to dual coordinates $\tilde{\sigma}_I \simeq \frac{\partial \mathcal{V}}{\partial \sigma^I}$ and the momenta are given by $p_I = \dot{\sigma}_I$. This observation should be useful when investigating non-extremal black hole solutions, where the constraint is deformed into $\mathcal{H} = c^2$. We leave a detailed investigation of non-extremal solutions to future work.

3. The dual picture

Given that we interpret the solutions we have constructed as instantons, we should expect that by substitution of the solution into the action we obtain a finite and positive result which is proportional to the instanton charges. But since the scalar fields vary along null directions of the target space it is clear that instanton action, when computed using (2.1) is identically zero. Thus the same feature which allows to have non-trivial instanton solutions renders their interpretation as instantons problematic. This is one aspect of the more fundamental problem of working with a Euclidean action which is not positive definite.

The same observations and questions apply to the type-IIB D-instanton solution [41] and other stringy instantons, such as instanton solutions for four-dimensional hypermultiplets [42, 44, 45, 46, 25]. For the purpose of generating higher-dimensional stationary solutions none of the above points is critical, except perhaps, that one might expect the ADM mass of black hole or other soliton to be related to the action of the instanton obtained by dimensional reduction with respect to time. For the D-instanton and various other similar instanton solutions it is known that one can obtain an instanton action which is finite, positive and proportional to the instanton charges by working with a dual version of the action (2.1), which is obtained by dualizing the axionic scalars b^I into tensor fields. Alternatively, a specific boundary term can be added to (2.1). In this section we derive the relevant formulae for the dualization of sigma models of the type (2.1). Later we will show that the resulting instanton actions agree with the masses of the solitons obtained by dimensional lifting.

In the sigma model (2.1), the axionic scalars b^I enter into the field equations only through their ‘field strength’ $F_m^I = \partial_m b^I$, which can be re-expressed in terms of the Hodge-dual three-forms $H_{mnp|I}$. By construction, the three-forms will satisfy the Bianchi identities

$$\partial_{[m} H_{npq]|I} = 0 , \quad (3.1)$$

and therefore they can be written, at least locally, as the exterior derivatives of two-form gauge fields $B_{mn|I}$. The standard Lagrangian for a theory of scalars σ^I and two-form gauge fields $B_{mn|I}$ takes the form

$$\mathcal{L} = -\frac{1}{2} N_{IJ}(\sigma) - \frac{1}{2 \cdot 3!} N^{IJ}(\sigma) H_{mnp|I} H_J^{mnp} , \quad (3.2)$$

where

$$H_{mnp|I} = 3! \partial_{[m} B_{np|I} ,$$

and where N^{IJ} is the inverse of N_{IJ} . Our parametrization anticipates that the dualization of antisymmetric tensor fields into axions inverts the coupling matrix. The Euclidean form of the Lagrangian is obtained by a Wick rotation, and the resulting Euclidean action

$$S_E[\sigma, B] = - \int d^4x \mathcal{L} \quad (3.3)$$

is positive definite.¹³ We will now show that this action is equivalent to (2.1), in the sense that it gives rise to the same equations of motion. The first step is to promote the Bianchi identity $\partial_{[m} H_{npq]|I} = 0$ to a field equation by introducing a Lagrange multiplier term:

$$S = \int d^4x \left(\frac{1}{2} N_{IJ}(\sigma) \partial_m \sigma^I \partial^m \sigma^J + \frac{1}{2 \cdot 3!} N^{IJ}(\sigma) H_{mnp|I} H_J^{mnp} + \lambda b^I \epsilon^{mnpq} \partial_m H_{npq|J} \right) . \quad (3.4)$$

¹³We include an explicit sign in this definition, so that the Euclidean action is positive definite instead of negative definite.

Here b^I is the Lagrange multiplier for the I -th Bianchi identity, and λ is a normalisation constant which we will fix to a convenient value later. Variation of this action with respect to H_I^{mnp} gives their equations of motion, which state that H_I^{mnp} and $\partial_m b^I$ are Hodge dual:

$$H_I^{mnp} = 3!\lambda N_{IJ}(\sigma)\epsilon^{mnpq}\partial_q b^J. \quad (3.5)$$

When we substitute this back into the action, we obtain

$$\begin{aligned} S[\sigma, b] = & \int d^4x \left(\frac{1}{2} N_{IJ}(\sigma) \partial_m \sigma^I \partial^m \sigma^J - \frac{1}{2} (3!\lambda)^2 N_{IJ} \partial_m b^I \partial^m b^J \right) \\ & + (3!\lambda)^2 \oint d^3 \Sigma^m b^I N_{IJ} \partial_m b^J. \end{aligned} \quad (3.6)$$

The boundary term in the second line results from an integration by parts. We observe that the bulk term matches with (2.1) if we choose $(3!\lambda)^2 = 1$.

The equations of motion for σ^I and $B_{mn|I}$ are obtained by variation of (3.2):

$$\begin{aligned} \partial^m (N_{IJ} \partial_m \sigma^J) &= \frac{1}{2} \partial_I N_{JK} \partial_m \sigma^J \partial^m \sigma^K \\ \partial^m (N^{IJ} H_{mnp|J}) &= 0. \end{aligned} \quad (3.7)$$

By construction, they are converted into (2.10) by substituting in equation (3.5).

The action (3.3) is positive definite, and we can obtain instanton solutions by applying the Bogomol'nyi trick, i.e. by rewriting the action as sum of perfect squares, plus a remainder:

$$\begin{aligned} S[\sigma, B] = & \int d^4x \left[\frac{1}{2} \left(\partial_m \sigma^I \mp \frac{1}{3!} N^{IJ} \epsilon_{mnpq} H_J^{npq} \right)^2 \right. \\ & \left. \pm \frac{1}{3!} \partial_m \sigma^I \epsilon^{mnpq} H_{npq|I} \right]. \end{aligned} \quad (3.8)$$

Note that the last term is a total derivative as a consequence of the Bianchi identity for $H_{mnp|I}$. In contrast to the similar rewriting (2.34) used in the previous section, this bulk term is not just an alternating sum of squares, but a single perfect square. Therefore equating the square to zero does not just give a saddle point, but a minimum of the action. The resulting equation

$$\partial_m \sigma^I = \pm \frac{1}{3!} N^{IJ} \epsilon_{mnpq} H_J^{npq}, \quad (3.9)$$

is the Hodge dual version of the extremal instanton ansatz (2.11), as we see immediately using (3.5). If the scalar metric N_{IJ} admits a non-trivial R -matrix (2.16), we can impose a Hodge dual version of the generalized instanton ansatz (2.17). As soon as we impose the (generalized) extremal instanton ansatz, the equations of motion (3.7) reduce to (2.12). Note that the dual instanton ansatz in combination with the Bianchi identity (3.1) already implies the equations of motion (2.12).

The extremal instanton ansatz is similar to the (anti-)selfduality condition characteristic for Yang-Mills instantons. This interesting feature is less obvious when working with the

purely scalar version (2.1) of the theory. To compute the instanton action, we substitute the relation (3.9) back into the action and obtain:

$$S_{\text{inst}} = \int d^4x N_{IJ} \partial_m \sigma^I \partial^m \sigma^J . \quad (3.10)$$

This is a boundary term, up to terms proportional to the equations of motion:

$$S_{\text{inst}} = \oint d^3\Sigma^m N_{IJ} \sigma^I \partial_m \sigma^J . \quad (3.11)$$

Guided by the analogy to Yang-Mills instantons, we expect that this can be expressed in terms of charges. The B -field has an abelian gauge symmetry, $B_{mn} \rightarrow B_{mn} + 2\partial_{[m}\Lambda_{n]}$, and one can define the associated electric and magnetic charges. For us the magnetic charges

$$Q_I = \frac{1}{3!} \oint d^3\Sigma^m \epsilon_{mnpq} H_I^{npq}$$

will be relevant. The normalization has been chosen such that they agree with the axionic charges (2.23) when we substitute (3.5). When evaluated on instanton solutions these charge take the form

$$Q_I = \oint d^3\Sigma^m N_{IJ} \partial_m \sigma^J .$$

Comparing this to the instanton action, we see that the instanton action takes the form

$$S_{\text{Inst}} = \sigma^I(\infty) Q_I , \quad (3.12)$$

provided that the boundary terms corresponding to the localized (-1) -branes (i.e. the centers of the harmonic functions) do not contribute. We will investigate this assumption below.

The boundary term obtained by dualizing the B -fields into axions b^I is

$$S_{\text{bd}} = \oint d^3\Sigma^m b^I N_{IJ} \partial_m b^J = b^I(\infty) \tilde{Q}_I , \quad (3.13)$$

where $\tilde{Q}_I = R_I{}^J Q_J$ and $\partial_m b^I = R^I{}_J \partial_m \sigma^J$. Thus the boundary action equals the instanton action when evaluated on instanton solutions, provided that $b^I(\infty) = R^I{}_J \sigma^J(\infty)$. Since the b^I are only defined up to constant shifts, we can regards this as a choice of gauge. This observation suggests to add the boundary action to the scalar bulk action (2.1), so that by evaluation on instanton solutions we obtain the same numerical values as for the scalar-tensor action.

Above we have made the assumption that the instanton solution is regular at the centers, and that the centers do not contribute to the instanton action. However, the contribution of a single center to the instanton action is

$$\lim_{r \rightarrow 0} \oint_{S_r^3} d^3\Sigma^m N_{IJ} \sigma^I \partial_m \sigma^J = \lim_{r \rightarrow 0} 2\pi^2 r^3 N_{IJ} \sigma^I \partial_r \sigma^J .$$

Since $N_{IJ} \partial_m \sigma^J$ is the derivative of a harmonic function, we know that close to a center

$$N_{IJ} \partial_r \sigma^J \sim \frac{1}{r^3} .$$

To have a finite contribution to the instanton action we must require that the scalars σ^I have finite limits at the centers. To obtain (3.12) we need to impose the stronger condition that the scalars σ^I vanish at the centers. The standard example of scalar instantons which we have in mind, including the D-instanton, have this property. Moreover, for supersymmetric models we expect a relation of the form (3.12) between the instanton action and a central charge of the supersymmetry algebra. Therefore we require that instanton solutions satisfy (3.12). Solutions which do not satisfy this condition should not be interpreted as proper instantons.

4. Finiteness of the instanton action and attractor behaviour. Examples

In this section we investigate the behaviour of instanton solutions. Our main interest is to find criteria which allow us to decide whether a given Hesse potential allows solutions with finite instanton action or not. This requires to investigate the behaviour of solutions at the centers, which in turn tells us whether solutions exhibit attractor behaviour, meaning that the asymptotics at the centers is determined exclusively by the charges, and in particular is independent of the boundary condition imposed at infinity. The fixed point behaviour of extremal black holes is a prototypical example, but we will encounter a slightly different behaviour which loosely speaking corresponds to fixed points ‘at infinity’. Later we will see that for those solutions that lift to five-dimensional black holes this behaviour is nevertheless equivalent to the (five-dimensional) black hole attractor mechanism. In order to obtain these results, we will need to make some assumptions about the Hesse potential in order to be able to control the asymptotic behaviour of solutions at the centers. Two types of Hesse potentials allow a complete analysis: homogeneous functions and logarithms of homogeneous functions. The second class corresponds to models which can be lifted to five dimensions in presence of gravity. We will also use this section to present a variety of explicit solutions.

4.1 Hesse potential $\mathcal{V} = \sigma^p$

We start with models where the Hesse potential depends on one single scalar σ and is homogeneous of degree $p = N + 2$, i.e. $\mathcal{V} \sim \sigma^{N+2}$. Then the metric is proportional to σ^N , and the sigma model takes the form

$$S = \frac{1}{2} \int d^4x \sigma^N (\partial_m \sigma \partial^m \sigma - \partial_m b \partial^m b) .$$

The case $N = 0, p = 2$ corresponds to a free theory. The case $N = 3, p = 1$ corresponds to Euclidean vector multiplets, obtained by temporal reduction of five-dimensional vector multiplets. We would like to include the case $N = -2$, which will turn out to be related to supergravity and, more generally, to models including gravity. Here the Hesse potential is not a homogeneous polynomial, but logarithmic, $\mathcal{V} = -\log \sigma$.¹⁴ For $N = -1$ the Hesse potential is the integral of the logarithm.

¹⁴Logarithmic Hesse potentials will be investigated in detail in Sections 4.6 – 4.8. In Section 6 we will present a modified formulation of the Hessian geometry of the target space, which is more suitable for this case.

By imposing the extremal instanton ansatz $\partial_m \sigma = \pm \partial_m b$, the equation of motion reduces to

$$\partial_m (\sigma^N \partial^m \sigma) = 0$$

which is equivalent to

$$\Delta \sigma^{N+1} = 0 .$$

In other words, σ^{N+1} is the dual coordinate of σ , which is of course a special case of the relation (2.22). Close to a center, the solution has the asymptotic form

$$\sigma^{N+1} \sim \frac{1}{r^2} ,$$

which implies that

$$\sigma \sim r^{\frac{-2}{N+1}} .$$

Consequently

$$\sigma \xrightarrow[r \rightarrow 0]{} \begin{cases} 0 & \text{if } N < -1 , \\ \infty & \text{if } N > -1 . \end{cases}$$

Therefore a finite action of the form $S_{\text{Inst}} = \sigma^I(\infty) Q_I$ is obtained for $N = -2, -3, \dots$, i.e. for logarithmic prepotentials and for prepotentials which are homogeneous of negative degrees $p = -1, -2, \dots$. For models with $N = 0, 1, 2, 3, \dots$ (i.e. with prepotentials homogeneous of degree $p = 2, 3, \dots$), the instanton action is infinite, due to contributions from the centers. Therefore these models do not possess proper (finite action) instanton solutions. This includes the case $N = 1, p = 3$, which corresponds to the temporal reduction of five-dimensional vector multiplets. The case $N = -1$, which is not covered by the above analysis, has to be treated separately. One finds that $\log \sigma$ is harmonic, and therefore the limit at a center is either zero or infinite, depending on the sign of the charge.

4.2 General homogeneous Hesse potentials

We now turn to Hesse potentials which depend on an arbitrary number of scalar fields, and are homogeneous of degree p . In this case the dual scalars

$$\sigma_I \simeq \mathcal{V}_I = \frac{\partial \mathcal{V}}{\partial \sigma^I}$$

are homogenous functions of degree $p-1$ of the scalars σ^I . Since $\Delta \sigma_I = 0$, the dual scalars have the asymptotics $\sigma_I \sim r^{-2}$ at the centers, implying that

$$\sigma^I \sim r^{-2/(p-1)} .$$

This is the natural generalization of the result obtained in the case of a single scalar: instanton solutions have a finite action of the form (3.12), if the Hesse potential is homogeneous of degree $p \leq -1$. We will come back to the case of logarithmic prepotentials later.

As the scalar fields always run off to either 0 or ∞ at the centers, we need to investigate whether these points are at finite or infinite ‘distance’. Since the scalar fields vary along isotropic submanifolds, the concept of distance has to be replaced by the concept of an

affine curve parameter. It is sufficient to consider single-centered solutions, and therefore we have to investigate whether the point $r = 0$ is at finite or infinite value of an affine parameter along the null geodesic corresponding to the solution. In terms of the dual scalars the equation of motion is always $\Delta\sigma_I = 0$, which, for single centered solutions, is the geodesic equation for a curve, with the radial variable r as curve parameter:

$$\Delta\sigma_I = \frac{\partial^2\sigma_I}{\partial r^2} + \frac{3}{r}\frac{\partial\sigma_I}{\partial r} = 0 .$$

Passing to an affine curve parameter

$$\tau = \frac{A}{r^2} + B$$

where $A \neq 0$ and B are constants, we obtain the affine version of the geodesic equation. Irrespective of the choice of affine parameter, we find that

$$\lim_{r \rightarrow 0} \tau(r) \longrightarrow \infty ,$$

which shows that the point $r = 0$ is at infinite affine parameter. Therefore the scalars always run away to limit points at ‘infinite distance’ on the scalar manifold. This is different from the fixed point behaviour observed for extremal black holes, where the scalars approach interior points of the scalar manifold, which are determined by the charges through the black hole attractor equations. However, for homogeneous prepotentials the run-away behaviour is not generic and shows features resembling fixed point behaviour. If we consider ratios of scalar fields, then the limits at the centers are finite and depend only on the charges

$$\frac{\sigma_I}{\sigma_J} \rightarrow \frac{q_I}{q_J} .$$

Thus at least the ratios show fixed point behaviour.

The asymptotic behaviour of the scalars at the centers can be represented alternatively by performing a (singular) rescaling, which brings the limit points to finite parameter values. One possible rescaling is to simply rescale the scalars according to

$$\hat{\sigma}_I := r^2 \sigma_I .$$

Then the new scalar $\hat{\sigma}_I$ show proper fixed point behaviour $\hat{\sigma}_I \rightarrow q_I$. A more intrinsic way of performing a rescaling is to divide the dual scalars σ_I by a homogeneous function of the scalars, which is chosen such that the new scalar fields are homogeneous of degree zero. The natural way of achieving this is to take the appropriate power of the Hesse potential:

$$\tilde{\sigma}_I = \frac{\sigma_I}{\mathcal{V}(\sigma)^{(p-1)/p}} \xrightarrow{r \rightarrow 0} \text{finite} ,$$

because

$$\sigma_I \sim \frac{1}{r^2} , \quad \sigma^I \sim \left(\frac{1}{r^2} \right)^{1/(p-1)} , \quad \mathcal{V}(\sigma) \sim \left(\frac{1}{r^2} \right)^{p/(p-1)} , \quad \mathcal{V}(\sigma)^{(p-1)/p} \sim \frac{1}{r^2} .$$

These rescalings have no immediate physical meaning, but are convenient for visualizing solutions. However, for models with logarithmic models the rescaling acquires a physical meaning once we couple the model to gravity, as we will see in Section 6. Although we did not yet discuss examples with logarithmic Hesse potential, it is clear by inspection that the above analysis remains valid for the corresponding $N = -2$ and $p = 0$.

4.3 Hesse potential $\mathcal{V} = \frac{1}{6}C_{IJK}\sigma^I\sigma^J\sigma^K$

If we construct models by temporal reduction of rigidly supersymmetric five-dimensional vector multiplets, then the most general Hesse potential is a cubic polynomial [36]. Since constant and linear terms do not enter into the metric, while quadratic terms only give a constant contribution to the scalar metric, we can restrict ourselves to homogeneous cubic polynomials

$$\mathcal{V}(\sigma) = \frac{1}{6}C_{IJK}\sigma^I\sigma^J\sigma^K.$$

The corresponding metric is¹⁵

$$N_{IJ} = \mathcal{V}_{IJ} = C_{IJK}\sigma^K.$$

The dual coordinates σ_I , for which the equations of motion reduce to $\Delta\sigma_I = 0$, are normalized according to

$$\sigma_I = \frac{1}{3}\mathcal{V}_I = \frac{1}{6}C_{IJK}\sigma^J\sigma^K = \frac{1}{6}N_{IJ}\sigma^K.$$

With this normalization

$$\sigma_I\sigma^I = \mathcal{V}(\sigma).$$

In terms of the dual coordinates, single and multi-centered solutions take the form

$$\sigma_I = h_I + \frac{q_I}{r^2}$$

and

$$\sigma_I = h_I + \sum_{a=1}^n \frac{q_{Ia}}{|x - x_a|^2}$$

respectively. In general, we cannot find an explicit expression for σ^I in terms of σ_I and, hence, in terms of the harmonic functions.

Hesse potential $\mathcal{V} = \sigma^1\sigma^2\sigma^3$

We now consider a special case where one can obtain explicit expressions for the σ^I . This model is closely related to the so-called STU-model. The Hesse potential is

$$\mathcal{V} = \sigma^1\sigma^2\sigma^3,$$

and the dual coordinates are chosen¹⁶

$$\sigma_1 = \sigma^2\sigma^3, \quad \sigma_2 = \sigma^3\sigma^1, \quad \sigma_3 = \sigma^1\sigma^2.$$

In terms of dual coordinates, the solution is

$$\sigma_I = H_I$$

¹⁵We use a notation where $\mathcal{V}_I = \frac{\partial\mathcal{V}}{\partial\sigma^I}$, $\mathcal{V}_{IJ} = \frac{\partial^2\mathcal{V}}{\partial\sigma^I\partial\sigma^J}$, etc.

¹⁶For convenience, we have changed the normalization of the σ_I compared to the case of a general cubic Hesse potential.

where H_I , $I = 1, 2, 3$ are harmonic functions. In this case we can solve explicitly for the σ^I :

$$\sigma^1 = \sqrt{\frac{\sigma_2 \sigma_3}{\sigma_1}} = \sqrt{\frac{H_2 H_3}{H_1}} ,$$

with similar expressions for σ^2, σ^3 obtained by cyclic permutations. Here we see explicitly that the fields σ^I diverge like $\frac{1}{r}$ for $r \rightarrow 0$, while their ratios are finite and only depend on the charges:

$$\frac{\sigma^1}{\sigma^2} = \frac{H_2}{H_1} \rightarrow \frac{q_2}{q_1} .$$

4.4 Hesse potential $\mathcal{V} = \frac{1}{4!} C_{IJKL} \sigma^I \sigma^J \sigma^K \sigma^L$

The next example is similar, but not extendable to a supersymmetric model. We take a general quartic Hesse potential

$$\mathcal{V} = \frac{1}{4!} C_{IJKL} \sigma^I \sigma^J \sigma^K \sigma^L .$$

The corresponding sigma model is still para-Kähler, but not special para-Kähler because the para-Kähler potential does not have a para-holomorphic prepotential. As a shortcut, we observe that the corresponding Euclidean sigma model lifts to a five-dimensional field theory whose couplings are encoded by a quartic Hesse potential. However five-dimensional supersymmetry requires a Hesse potential which is at most cubic.

The corresponding metric is

$$N_{IJ} = \frac{1}{2} C_{IJKL} \sigma^K \sigma^L ,$$

and dual coordinates are given by

$$\sigma_I = \frac{1}{4!} C_{IJKL} \sigma^J \sigma^K \sigma^L = \frac{1}{6} \mathcal{V}_I .$$

The solution is given in terms of harmonic functions by $\sigma_I = H_I$. While we cannot solve for the σ^I explicitly, homogeneity implies that the $\sigma^I \sim r^{-2/3}$ for $r \rightarrow 0$, and that the ratios $\frac{\sigma_I}{\sigma_J}$ and $\frac{\sigma^I}{\sigma^J}$ have finite limits.

Explicit solutions can be obtained for sufficiently simple choices of a quartic Hesse potential, for example

$$\mathcal{V} = \sigma^1 \sigma^2 \sigma^3 \sigma^4 .$$

Normalizing the dual coordinates such that

$$\sigma_1 = \sigma^2 \sigma^3 \sigma^4 , \dots$$

the solution is

$$\sigma^1 = \left(\frac{\sigma_2 \sigma_3 \sigma_4}{(\sigma_1)^2} \right)^{1/3} = \left(\frac{H_2 H_3 H_4}{H_1^2} \right)^{1/3} , \dots$$

with similar expressions for the other σ^I obtained by cyclic permutations.

4.5 Hesse potential $\mathcal{V} = -\log(\sigma)$

In the following sections we discuss models with logarithmic Hesse potentials. As we will see in Section 6 these are the models which can be lifted to five-dimensional Einstein-Maxwell type theories. We will study some aspects already here, because these models can as well be lifted to five dimension without coupling to gravity.

We start with a logarithmic Hesse potential depending on a single scalar,

$$\mathcal{V} = -\log \sigma$$

where $\sigma > 0$. The resulting Hessian metric is

$$\mathcal{V}'' = \frac{1}{\sigma^2}.$$

We have already seen that this model is in the class where the instanton action has the form (3.12). The dual coordinate is proportional to \mathcal{V}' , and we normalize it to be $\frac{1}{\sigma}$. The reduced equation of motion is

$$\Delta \frac{1}{\sigma} = 0,$$

which is solved by

$$\sigma = \frac{1}{H},$$

where H is a harmonic function. Considering a single centered solution,

$$\sigma = \frac{1}{h + \frac{q}{r^2}},$$

we can see explicitly how σ behaves for $r \rightarrow \infty$ and $r \rightarrow 0$:

$$\sigma \xrightarrow{r \rightarrow \infty} \frac{1}{h}, \quad \sigma \xrightarrow{r \rightarrow 0} 0.$$

This illustrates our general result, and we can see explicitly that the action is finite.

The target space corresponding to this model is the symmetric space $SL(2, \mathbb{R})/SO(1, 1)$, which is also known as AdS^2 . The action, expressed in terms of the scalars σ and b is

$$S = \int d^4x \frac{1}{\sigma^2} (\partial_m \sigma \partial^m \sigma - \partial_m b \partial^m b).$$

In terms of the para-complex coordinates $X = \sigma + eb$ this becomes

$$S = \int d^4x \frac{\partial_m X \partial^m \bar{X}}{(\text{Re}(X))^2},$$

which makes explicit that the target space is a para-Kähler manifold with para-Kähler potential

$$K = -\log(X + \bar{X}).$$

By the analytic continuation $b \rightarrow ib$ we obtain the upper half plane, equipped with the Poincaré metric, $\mathcal{H} \cong \frac{SL(2, \mathbb{R})}{SO(2)}$.¹⁷

¹⁷Various coordinate systems for the two symmetric spaces in question can be found, for example, in [55].

4.6 Hesse potential $\mathcal{V} = -\log(\sigma^1\sigma^2\sigma^3)$

Another model, which turns out to be the Euclidean version of the well-known STU-model is obtained by taking three copies of the previous model. The Hesse potential is

$$\mathcal{V} = -\log(\sigma^1\sigma^2\sigma^3) = -\log\sigma^1 - \log\sigma^2 - \log\sigma^3.$$

The corresponding target space is the product of three copies of $SL(2, \mathbb{R})/SO(1, 1)$, which is para-Kähler with para-Kähler potential

$$K = -\log((X^1 + \bar{X}^1)(X^2 + \bar{X}^2)(X^3 + \bar{X}^3)),$$

where $X^I = \sigma^I + e b^I$. This target space is in fact even projective special para-Kähler, with para-holomorphic prepotential $F = -\frac{X^1 X^2 X^3}{X^0}$, as it must be for Euclidean vector multiplets coupled to supergravity [38].

The dual coordinates can be normalized to be

$$\sigma_I = \frac{1}{\sigma^I},$$

so that explicit solutions for the σ^I can be found:

$$\sigma^I = \frac{1}{H_I}.$$

We will see later that this solution can be lifted to a five-dimensional extremal black hole solution of five-dimensional supergravity.

4.7 Hesse potential $\mathcal{V} = -\log \hat{\mathcal{V}}(\sigma)$, with homogeneous $\hat{\mathcal{V}}(\sigma)$

Finally, we consider the general case of a Hesse potential which is the logarithm of a homogeneous function $\hat{\mathcal{V}}(\sigma)$ (of arbitrary degree):

$$\mathcal{V}(\sigma^I) = -\log \hat{\mathcal{V}}(\sigma^I)$$

where

$$\hat{\mathcal{V}}(\lambda\sigma^I) = \lambda^p \hat{\mathcal{V}}(\sigma^I)$$

with integer p . Then the Hesse potential is not strictly a homogeneous function, but it is homogeneous of degree zero up to a constant shift. However, the first derivatives

$$\sigma_I \simeq \frac{\partial \mathcal{V}}{\partial \sigma^I}$$

are homogeneous of degree -1 , and the metric, which is given by the second derivatives,

$$N_{IJ} = \frac{\partial^2 \mathcal{V}}{\partial \sigma^I \partial \sigma^J}$$

is homogeneous of degree -2 . This corresponds to the case $N = -2$ discussed in Sections 4.1 and 4.2, and the results derived there apply (setting $N = -2$ and $p = 0$ in the relevant formulae). In particular the instanton action is of the form (3.12), and the solutions show a version of fixed point behaviour where the scalars run off to a point at infinite affine parameter while the ratios approach finite values determined by the charges.

5. Lifting to five dimensions, without gravity

In the following section we discuss the lifting of instanton solutions to five-dimensional solitons in the absence of gravity. Here no constraints need to be imposed on the Hesse potential. We show that the mass of the soliton obtained by lifting is equal to the instanton action.

The para-Hermitean Euclidean sigma model (2.1) can be lifted to a five-dimensional theory of scalars and gauge fields:

$$S[\sigma, A_\mu] = \int d^5x \left(-\frac{1}{2} N_{IJ}(\sigma) \partial_\mu \sigma^I \partial^\mu \sigma^J - \frac{1}{4} N_{IJ}(\sigma) F_{\mu\nu}^I F^{\mu\nu|J} + \dots \right). \quad (5.1)$$

Here $\mu, \nu = 0, 1, \dots, 4$ are five-dimensional Lorentz indices, and the four-dimensional axions have been identified with the time components of the five-dimensional gauge fields

$$b^I = -A_0^I.$$

To obtain a covariant theory, we have added the magnetic components F_{mn}^I , $m, n = 1, \dots, 4$ of the five-dimensional field strength. We also allow further terms, as long as they do not contribute to the four-dimensional sigma model obtained by reduction over time. It is straightforward to verify that the five-dimensional action (5.1) reduces to the para-Hermitean sigma model (2.1) upon restricting to static and purely electric field configurations, and reducing with respect to time. Thus instanton solutions of (2.1) lift to electrically charged solitons of (5.1).

The full field equations of the five-dimensional theory have the following form. The equation of motion for the scalars σ^I is

$$N_{KJ} \square \sigma^J + \frac{1}{2} \partial_K N_{IJ} \partial_\mu \sigma^I \partial^\mu \sigma^J = \frac{1}{4} \partial_K N_{IJ} F_{\mu\nu}^I F^{\mu\nu|J},$$

and the equation of motion of the five-dimensional gauge fields is

$$\partial_\mu (N_{IJ} F^{\mu\nu|J}) = 0.$$

If we impose that the solution is static and does not carry magnetic charge, then all time-derivatives vanish and the only non-vanishing field strength components can be expressed in terms of the electrostatic potentials A_0^I :

$$F_{tm} = -F_{mt} = -\partial_m A_0^I = \partial_m b^I.$$

In such backgrounds the equations of motion take the following form:

$$\begin{aligned} N_{KJ} \Delta \sigma^J + \frac{1}{2} \partial_K N_{IJ} \partial_m \sigma^I \partial^m \sigma^J &= \frac{1}{2} \partial_K N_{IJ} F_{0m}^I F^{0m|J} \\ \partial_m (N_{IJ} \partial^m A_0^J) &= 0. \end{aligned} \quad (5.2)$$

Expressing F_{0m}^I and A_0^I in terms of b^I , we see that these equations of motion are identical to (2.10). The extremal instanton ansatz corresponds to imposing

$$\partial_m \sigma^I = \pm F_{0m}^I$$

which means that the scalars σ^I are proportional to the electrostatic potentials. For five-dimensional vector multiplets, this is the condition for a BPS solution supported by scalars and electric fields.

Imposing the extremal instanton ansatz we therefore obtain the reduced equations of motion

$$\partial_m(N_{IJ}\partial^m\sigma^J) = 0 ,$$

which is identical to (2.12). The four-dimensional instanton charges equal the five-dimensional electric charges, which are defined by

$$Q_I = \oint d^3\Sigma^m N_{IJ} F_{0m}^J = \oint d^3\Sigma^m N_{IJ} \partial_m b^J .$$

From the five-dimensional point of view the method used in Section 2.3 to solve the equations of motion is a standard method for solving Maxwell-type equations in an electrostatic background.

We expect that the four-dimensional instanton action is related to the five-dimensional mass. The mass of the soliton is obtained by integrating the energy density, which is the component T_{00} of the energy momentum tensor $T_{\mu\nu}$, over space. We use the symmetric energy momentum tensor which is obtained by coupling the action (5.1) to a background metric and varying it. The result is

$$\begin{aligned} T_{\mu\nu} = & N_{IJ} \partial_\mu \sigma^I \partial_\nu \sigma^J - \frac{1}{2} N_{IJ} \eta_{\mu\nu} \partial_\rho \sigma^I \partial^\rho \sigma^J \\ & + N_{IJ} F_{\mu\rho}^I F_\nu^{\rho J} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma}^I F^{\rho\sigma J} . \end{aligned} \quad (5.3)$$

In a static, purely electric background, the resulting energy density is

$$T_{00} = \frac{1}{2} N_{IJ} \partial_m \sigma^I \partial^m \sigma^J + \frac{1}{2} N_{IJ} \delta^{mn} F_{0m}^I F_{0n}^J .$$

For solutions where $F_{0m}^I = \pm \partial_m \sigma^I$, this becomes

$$T_{00} = N_{IJ} \partial_m \sigma^I \partial^m \sigma^J .$$

The integral expression for the soliton mass M of the soliton agrees with the instanton action (3.10) and the boundary action (3.13):

$$M = \int d^4x T_{00} = S_{\text{inst.}} = S_{\text{bound.}} .$$

Our previous discussion of fixed point behaviour of the scalars σ^I remains valid, because there is no difference between the four- and five-dimensional scalars. In particular the soliton mass is finite if the σ^I approach finite values, and it is given in terms of the five-dimensional electric charges as

$$M = \sigma^I(\infty) Q_I ,$$

if the scalars go to zero at the centers. Models where the Hesse potential is homogeneous of positive degree do not have proper, i.e. finite mass, solitons of the type considered. This includes rigid five-dimensional vector multiplets. If the degree of homogeneity is negative, or if the Hesse potential is the logarithm of a homogeneous function, then solitons with finite mass do exist.

6. Lifting to five dimensions, with gravity

6.1 Dimensional lifting and dimensional reduction

We now turn to the lifting of four-dimensional instantons to five-dimensional black holes. In the presence of gravity the relation between the five-dimensional and four-dimensional actions becomes more complicated. As a first step we would like to identify the class of five-dimensional Einstein-Maxwell type actions which reduce to actions of the form (2.15) by dimensional reduction with respect to time. To be precise we allow additional terms in both actions, as long as (2.15) is a consistent reduction, i.e., as long as solutions of (2.15) are solutions of the five-dimensional theory. The main new feature in the presence of gravity is that the decomposition of the five-dimensional metric gives rise to a Kaluza-Klein scalar and Kaluza-Klein gauge field. The Kaluza-Klein gauge field can be set to zero consistently. At the level of five-dimensional solutions this means that we restrict ourselves to solutions which are not only stationary, but static, i.e. we exclude rotating solutions. However the Kaluza-Klein scalar provides a complication because it needs to be incorporated into the four-dimensional scalar sigma model.¹⁸

One class of examples where one obtains Euclidean actions of the type (2.15) is the temporal reduction of five-dimensional supergravity coupled to vector multiplets [38]. We will adopt the strategy of generalizing this class while keeping the relevant feature that temporal reduction gives rise to a para-Kähler sigma model. As far as the relation between five-dimensional and four-dimensional actions is concerned, the analysis can be carried out in parallel for spatial and temporal reduction. For concreteness we will take the case of temporal reduction, but the other case is simply obtained by flipping signs in the Lagrangian, as discussed in more detail in [38].

The geometry underlying five-dimensional supergravity with vector multiplets is the local (or projective) version of very special real geometry [35]. This is a type of Hessian geometry, where the Hesse potential $\mathcal{V} = -\log \hat{\mathcal{V}}$ is the logarithm of a homogeneous cubic polynomial $\hat{\mathcal{V}}$, which is called the prepotential. To be precise, the metric obtained from this Hesse potential gives the coupling matrix of the gauge fields, while the scalar metric is its pull back to the hypersurface $\hat{\mathcal{V}} = 1$. This reflects that the supergravity theory has one scalar field less than it has gauge fields. A five-dimensional vector multiplet contains a gauge field and a real scalar, but the gravity multiplet contains an additional gauge field, the graviphoton. Upon dimensional reduction each gauge field gives rise to an axionic scalar, which can be combined with the five-dimensional scalars and the Kaluza-Klein scalar into a sigma model of the type (2.15). Thus it is important to have one additional gauge field in five dimensions. The other critical feature is that the metric of the five-dimensional scalar sigma model is homogeneous of degree -2 in the scalar fields. As we will see later this is crucial for combining the Kaluza-Klein scalar with the five-dimensional scalars in such a way that we obtain a sigma model of the form (2.15).

As we have seen in Section 4, a Hessian metric is homogeneous of degree -2 if its Hesse potential $\mathcal{V} = -\log \hat{\mathcal{V}}$ is the logarithm of a homogeneous function $\hat{\mathcal{V}}$, irrespective of the

¹⁸As will be explicit from the solutions discussed later, freezing the Kaluza-Klein scalar is not an option, since this would only leave us with trivial solutions.

degree of homogeneity. Therefore we will generalize the local very special real geometry of supergravity by dropping the requirement that the prepotential $\hat{\mathcal{V}}$ is a homogeneous cubic polynomial, while still requiring that it is a homogeneous function of degree p , where p is now arbitrary.

Dimensional reduction of five-dimensional supergravity with vector multiplets with respect to space results in target space geometries which are projective special Kähler [35]. The map between the target geometries of five-dimensional and four-dimensional vector multiplets is the r -map [50], which we will call the local (or projective) r -map, to distinguish it from its rigid (or global) counterpart. If one reduces over time, one obtains projective special para-Kähler manifolds, and the corresponding map is called the local (or projective) para- r -map [38]. The following construction provides a generalization of both the local r -map and local para- r -map. For concreteness we will give explicit expression for the para- r -map, and explain in the end how the r -map is obtained by analytical continuation.

The construction starts with $n + 1$ scalar fields $h = (h^I) = (h^0, h^1, \dots, h^n)$, which we interpret as affine coordinates on an $(n + 1)$ -dimensional Hessian manifold \tilde{M}_r . We work locally and take \tilde{M}_r to be an open domain in \mathbb{R}^{n+1} . The Hesse potential for this manifold (which will be the prepotential for the actual scalar manifold M_r) is $\mathcal{V}(h) = -\log \hat{\mathcal{V}}(h)$, where the prepotential $\hat{\mathcal{V}}(h)$ is homogeneous of degree p :

$$\hat{\mathcal{V}}(\lambda h^0, \dots, \lambda h^n) = \lambda^p \hat{\mathcal{V}}(h^0, h^1, \dots, h^n) . \quad (6.1)$$

Taking the derivative with respect to λ we obtain

$$\hat{\mathcal{V}}_I(\lambda h) h^I = p \lambda^{p-1} \hat{\mathcal{V}}(h) , \quad (6.2)$$

where the subscript I denotes differentiation with respect to h^I . By setting $\lambda = 1$ we obtain

$$\hat{\mathcal{V}}_I h^I = p \hat{\mathcal{V}}(h) . \quad (6.3)$$

Further differentiation implies that

$$\hat{\mathcal{V}}_{IJ} h^I = (p - 1) \hat{\mathcal{V}}_J . \quad (6.4)$$

The logarithm of $\hat{\mathcal{V}}(h)$ is used to define a Hessian metric by

$$a_{IJ}(h) = -\frac{1}{p} \frac{\partial^2 \log \hat{\mathcal{V}}(h)}{\partial h^I \partial h^J} = -\frac{1}{p} \left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2} \right) .$$

A conventional factor $\frac{1}{p}$ has been introduced in order to be consistent with supergravity conventions for $p = 3$. The metric is homogeneous of degree -2 in the h^I . In order to ensure that the metric $a_{IJ}(h)$ is positive definite, we might need to restrict the fields $h = (h^I)$ to a suitable domain $D \subset \mathbb{R}^{n+1}$. The scalar target manifold M_r of the model is the hypersurface $\{h^I | \hat{\mathcal{V}}(h) = 1\}$ of D , equipped with the pull-back metric.

$$a_{xy}(\phi) = \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} a_{IJ}(h(\phi)) .$$

The physical scalars ϕ^x , $x = 1, \dots, n$ provide local coordinates on the hypersurface $\{h^I | \hat{\mathcal{V}} = 1\} \subset D$.

In the following it will be convenient to work with the fields h^I , which are subject to the constraint $\hat{\mathcal{V}}(h) = 1$, and with the associated Hessian metric $a_{IJ}(h)$. We will need a few relations involving $a_{IJ}(h)$. First note that (6.3) and (6.4) can be used to show that

$$\begin{aligned} a_{IJ}(h)h^I h^J &= -\frac{1}{p} \partial_I \partial_J \log \hat{\mathcal{V}}(h) h^I h^J = -\frac{1}{p} \left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2} \right) h^I h^J \\ &= \frac{\hat{\mathcal{V}}_J h^J}{p \hat{\mathcal{V}}} = 1 . \end{aligned} \quad (6.5)$$

Differentiation of the constraint $\hat{\mathcal{V}}(h) = 1$ with respect to space-time implies

$$\hat{\mathcal{V}}_I \partial_\mu h^I = 0 , \quad (6.6)$$

where $\mu = 0, \dots, 4$ are five-dimensional space-time indices. Combining this with (6.5) we obtain

$$a_{IJ} h^I \partial_\mu h^J = -\frac{\hat{\mathcal{V}}_J}{\hat{\mathcal{V}}} \partial_\mu h^J = 0 . \quad (6.7)$$

We now use the prepotential $\hat{\mathcal{V}}(h)$ to define the following five-dimensional bosonic Lagrangian:

$$e^{\hat{e}} \hat{\mathcal{L}} = \frac{\hat{R}}{2} - \frac{3}{4} a_{IJ}(h) \partial_\mu h^I \partial^\mu h^J - \frac{1}{4} a_{IJ}(h) F_{\mu\nu}^I F^{\mu\nu J} + \dots . \quad (6.8)$$

Here \hat{R} is the five-dimensional Ricci scalar, \hat{e} is the determinant of the local frame ('fünfbein'), $a_{IJ}(h)$ is the Hessian metric defined above, and for the scalar term the constraint $\hat{\mathcal{V}}(h) = 1$ is understood. As indicated, the Lagrangian might contain further terms, provided that these do not contribute to four-dimensional Euclidean sigma model obtained by reduction with respect to time. For $p = 3$, (6.8) is part of the Lagrangian of five-dimensional vector multiplets coupled to n vector multiplets. The full supergravity Lagrangian also contains a Chern-Simon terms and fermionic terms, which, however, do not contribute to the four-dimensional sigma model upon reduction.

We now reduce the Lagrangian (6.8) with respect to time.¹⁹ The reduction of the metric is carried out in such a way that the resulting four-dimensional Einstein-Hilbert term has the canonical form, i.e., we reduce from the five-dimensional Einstein frame to the four-dimensional Einstein frame. The corresponding parametrization of the line element is

$$ds_{(5)}^2 = -e^{2\tilde{\sigma}} (dt + \mathcal{A}_m dx^m)^2 + e^{-\tilde{\sigma}} ds_{(4)}^2 ,$$

where $\tilde{\sigma}$ is the Kaluza-Klein scalar and \mathcal{A}_m is the Kaluza-Klein vector. Upon dimensional reduction over time, the zero components \mathcal{A}_0^I of the five-dimensional gauge fields become four-dimensional scalar fields $m^I = \mathcal{A}_0^I$. In four dimensions, we only keep the Einstein-Hilbert term and the scalar terms. This is a consistent truncation and corresponds to the

¹⁹We refer to [38] for a more detailed discussion of dimensional reduction.

restriction to five-dimensional field configurations which are static and purely electric. The relevant part of the reduced Lagrangian is

$$e^{-1}\mathcal{L} = \frac{R}{2} - \frac{3}{4}\partial_m\tilde{\sigma}\partial^m\tilde{\sigma} - \frac{3}{4}a_{IJ}(h)\partial_m h^I\partial^m h^J + \frac{1}{2}e^{-2\tilde{\sigma}}a_{IJ}(h)\partial_m m^I\partial^m m^J, \quad (6.9)$$

where $m = 1, \dots, 4$ are indices in four-dimensional space, R is the four-dimensional Ricci scalar, and e is the determinant of the four-dimensional local frame ('vierbein'). By making the redefinitions

$$h^I = Ae^{-\tilde{\sigma}}\sigma^I, \quad (6.10)$$

$$m^I = Bb^I, \quad (6.11)$$

where A, B are constants to be fixed later, the Lagrangian takes on the form

$$\begin{aligned} e^{-1}\mathcal{L} = & \frac{R}{2} - \frac{3}{4}\partial_m\tilde{\sigma}\partial^m\tilde{\sigma} - \frac{3}{4}a_{IJ}(e^{-\tilde{\sigma}}\sigma)\sigma^I\sigma^J\partial_m e^{-\tilde{\sigma}}\partial^m e^{-\tilde{\sigma}} \\ & - \frac{3}{4}a_{IJ}(e^{-\tilde{\sigma}}\sigma)e^{-2\tilde{\sigma}}\partial_m\sigma^I\partial^m\sigma^J - \frac{3}{2}a_{IJ}(e^{-\tilde{\sigma}}\sigma)e^{-\tilde{\sigma}}\sigma^I\partial_m e^{-\tilde{\sigma}}\partial^m\sigma^J \\ & + \frac{B^2}{2A^2}e^{-2\tilde{\sigma}}a_{IJ}(e^{-\tilde{\sigma}}\sigma)\partial_m b^I\partial^m b^J. \end{aligned} \quad (6.12)$$

From now on we regard σ^I and b^I as the independent fields. Note that the constraint $\hat{\mathcal{V}}(h) = 1$ implies the relation

$$\hat{\mathcal{V}}(\sigma) = \hat{\mathcal{V}}(A^{-1}e^{\tilde{\sigma}}h) = A^{-p}e^{p\tilde{\sigma}}\hat{\mathcal{V}}(h) = A^{-p}e^{p\tilde{\sigma}}, \quad (6.13)$$

which expresses the Kaluza-Klein scalar $\tilde{\sigma}$ as a function of the four-dimensional scalars σ^I .

By considering the relations (6.5) and (6.7), the first and second term cancel and the fourth term vanishes. If we choose the constants A, B to satisfy

$$B^2 = \frac{3A^2}{2}, \quad (6.14)$$

and use that $a_{IJ}(h)$ is homogeneous of degree -2 , the remaining terms in the Lagrangian take the form

$$e^{-1}\mathcal{L} = \frac{R}{2} - \frac{3}{4}a_{IJ}(\sigma)\partial_m\sigma^I\partial^m\sigma^J + \frac{3}{4}a_{IJ}(\sigma)\partial_m b^I\partial^m b^J. \quad (6.15)$$

Defining

$$N_{IJ}(\sigma) = \frac{3}{2}a_{IJ}(\sigma), \quad (6.16)$$

we recognize the standard form (2.15) of a para-Hermitian sigma model with n commuting shift isometries, coupled to gravity,

$$e^{-1}\mathcal{L} = \frac{R}{2} - \frac{1}{2}N_{IJ}(\sigma)(\partial_m\sigma^I\partial^m\sigma^J - \partial_m b^I\partial^m b^J). \quad (6.17)$$

The metric $N_{IJ}(\sigma)$ has the Hesse potential $\mathcal{V}(\sigma) = -\log \hat{\mathcal{V}}(\sigma)$:

$$N_{IJ}(\sigma) = -\frac{3}{2p}\frac{\partial^2}{\partial\sigma^I\partial\sigma^J}\log \hat{\mathcal{V}}(\sigma).$$

As a result, the metric $N_{IJ} \oplus (-N_{IJ})$ of the scalar manifold spanned by σ^I, b^I is para-Kähler. This is seen explicitly by introducing para-holomorphic coordinates

$$X^I = \sigma^I + eb^I,$$

and computing

$$\frac{\partial^2 \log \hat{\mathcal{V}}}{\partial X^I \partial \bar{X}^J} = \frac{\partial^2 \log \hat{\mathcal{V}}}{\partial \sigma^K \partial \sigma^L} \frac{\partial \sigma^K}{\partial X^I} \frac{\partial \sigma^L}{\partial \bar{X}^J} = \frac{1}{4} \frac{\partial^2 \log \hat{\mathcal{V}}}{\partial \sigma^I \partial \sigma^J} = \frac{p}{6} N_{IJ}.$$

Thus $K(X, \bar{X}) = \frac{6}{p} \log \hat{\mathcal{V}}$ is a para-Kähler potential for the metric $N_{IJ} \oplus (-N_{IJ})$.

The relation between the five- and four-dimensional Lagrangian is true irrespective of the value of p that we choose, and hence it makes sense for models with $p \neq 3$, which cannot be embedded into a five-dimensional supersymmetric model. However, it was crucial that we could combine the Kaluza-Klein scalar with the five-dimensional scalars h^I in such a way that the scalar target manifold of the reduced theory became para-Hermitian. This worked only because the metric $a_{IJ}(h)$ is homogeneous of degree -2 . Therefore, there is no obvious further generalization which would allow one to drop the condition that the prepotential is homogeneous. The effect of reducing over space rather than time is to replace b^I by ib^I in 6.17. Equivalently, in terms of (para-)complex coordinates, it corresponds to replacing $X^I = \sigma^I + eb^I$ by $Y^I = \sigma^I + ib^I$, i.e. the para-complex structure is replaced by a complex structure, and one obtains a Kähler manifold where the Kähler potential is proportional to the prepotential $\hat{\mathcal{V}}(\sigma)$. Thus, as in [38] the para- r -map and r -map are related by analytic continuation (see also Appendix A).

Having fixed the relation between the five-dimensional and the four-dimensional theory, we can now see how four-dimensional instantons lift to five-dimensional solutions. We have restricted ourselves to solutions of (2.15) where the four-dimensional metric is flat, $ds_{(4)}^2 = \delta_{mn} dx^m dx^n$. Such line elements lift to five-dimensional line elements of the form

$$ds_{(5)}^2 = -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} \delta_{mn} dx^m dx^n,$$

where $\tilde{\sigma}$ is the Kaluza-Klein scalar. This is precisely the structure of a line element for an extremal five-dimensional black hole. Extremal black holes have the particular feature that their line elements reduce under temporal reduction to flat line elements, provided that one uses the Einstein frame in both dimensions. The non-trivial five-dimensional geometry is fully captured by the Kaluza-Klein scalar, while the four-dimensional metric is flat. This explains why extremal black holes correspond to null geodesics, and why we could effectively drop the four-dimensional Einstein-Hilbert term when constructing solutions. This observation provides additional justification for calling the corresponding instanton solutions extremal.

From the four-dimensional point of view all information is encoded in the scalar fields σ^I . With the choice $A = 1$, which implies $B = \sqrt{\frac{3}{2}}$, the Kaluza-Klein scalar is determined by the four-dimensional scalars through the relation

$$e^{p\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma), \tag{6.18}$$

while the five-dimensional scalars are given by

$$h^I = e^{-\tilde{\sigma}} \sigma^I ,$$

We have a Hesse potential of the form $\mathcal{V}(\sigma) = -\log \hat{\mathcal{V}}(\sigma)$, and therefore the dual scalars have the form

$$\sigma_I \simeq \frac{\partial}{\partial \sigma^I} \log \hat{\mathcal{V}}(\sigma) .$$

As in previous examples we will fix the factor of proportionality at our convenience. The solution is given by $\sigma_I(x) = H_I(x)$, where $H_I(x)$ are harmonic functions on \mathbb{R}^4 . Explicit expressions for the σ^I can only be obtained case by case if the prepotential is sufficiently simple. However, the asymptotics of the solution at the center is known from Section 4, and we will see below that this allows us to obtain information about the ADM mass and about the black hole entropy. The axions b^I are determined by the extremal instanton ansatz and in turn determine the five-dimensional gauge fields.

6.2 ADM mass and instanton action

Before looking into explicit examples, we show that the ADM mass of the five-dimensional black hole is equal to the action of the corresponding four-dimensional instanton. The ADM mass can be written as a surface integral involving the Kaluza-Klein scalar $\tilde{\sigma}$ [38]. To compare this to the instanton action, we express the ADM mass in terms of the prepotential:

$$M_{ADM} = -\frac{3}{2} \oint d^3 \Sigma^m \partial_m e^{-\tilde{\sigma}} = -\frac{3}{2} \oint d^3 \Sigma^m \partial_m \hat{\mathcal{V}}(\sigma)^{-1/p} .$$

Now we compare this to the instanton action

$$S_{\text{inst}} = \oint d^3 \Sigma^m N_{IJ} \sigma^I \partial_m \sigma^J .$$

The metric N_{IJ} is given by

$$N_{IJ} = -\frac{3}{2p} \left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2} \right) .$$

Using that $\hat{\mathcal{V}}(\sigma)$ is homogeneous of degree p , we find

$$N_{IJ} \sigma^I \partial_m \sigma^J = -\frac{3}{2p} \left(\frac{\hat{\mathcal{V}}_{IJ} \sigma^I}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \sigma_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2} \right) \partial_m \sigma^J = \frac{3}{2p} \frac{\hat{\mathcal{V}}_J}{\hat{\mathcal{V}}} \partial_m \sigma^J$$

But this is a total derivative:

$$N_{IJ} \sigma^I \partial_m \sigma^J = \frac{3}{2p} \partial_m \log \hat{\mathcal{V}}(\sigma) .$$

As a result we have

$$\begin{aligned} M_{ADM} &= -\frac{3}{2} \oint d^3 \Sigma^m \partial_m \hat{\mathcal{V}}(\sigma)^{-1/p} = -\frac{3}{2} \oint d^3 \Sigma^m \partial_m e^{-\tilde{\sigma}} , \\ S_{\text{inst}} &= \frac{3}{2} \oint d^3 \Sigma^m \partial_m \log \hat{\mathcal{V}}(\sigma)^{1/p} = \frac{3}{2} \oint d^3 \Sigma^m \partial_m \tilde{\sigma} . \end{aligned} \tag{6.19}$$

Both the ADM mass and the instanton action are surface integrals, but the integrands are different. To compare the integrals we rewrite the ADM mass as

$$M_{ADM} = \frac{3}{2} \oint d^3 \Sigma^m e^{-\tilde{\sigma}} \partial_m \tilde{\sigma} . \quad (6.20)$$

The integration is performed by integrating over a three-sphere of radius r and taking $r \rightarrow \infty$. Therefore the only terms in the integrand which give a finite contribute are those which fall off like $\frac{1}{r^3}$. The behaviour of the integrands in this limit is obtained by observing that $\hat{\mathcal{V}}(\sigma)$, and, hence, $e^{\tilde{\sigma}}$ are algebraic functions of the harmonic functions H_I . Since we normalize the five-dimensional metric to approach the standard Minkowski metric at infinity, both expressions approach the constant value 1 at infinity. This implies the following Taylor expansion of $\hat{\mathcal{V}}(\sigma)$ around $\tau = \frac{1}{r^2} = 0$:

$$\begin{aligned} \hat{\mathcal{V}}(\sigma) &= 1 + \mathcal{O}\left(\frac{1}{r^2}\right) . \\ \partial_m \hat{\mathcal{V}}(\sigma) &= \mathcal{O}\left(\frac{1}{r^3}\right) . \end{aligned}$$

This in turn implies that

$$\begin{aligned} e^{-\tilde{\sigma}} &= 1 + \mathcal{O}\left(\frac{1}{r^2}\right) , \\ \partial_m \tilde{\sigma} &= \mathcal{O}\left(\frac{1}{r^3}\right) . \end{aligned}$$

As a consequence, the factor $e^{-\tilde{\sigma}}$ in (6.20) does not contribute to the integral, and the ADM mass and the instanton agree,

$$M_{ADM} = S_{\text{Inst}} ,$$

despite that the integrands of the surface integrals are different.²⁰ This is the same result as we found when lifting without coupling to gravity. In absence of gravity mass is defined as the integral of the energy density, but no such definition is available in the presence of gravity. Instead one needs to apply the ADM definition of mass. The fact that we find agreement between mass and instanton action in both cases provides additional support for the definition of the instanton action obtained by dualization of axions into tensors.

6.3 Black hole entropy and the size of the throat

Besides the ADM mass, the black hole entropy is the most important property of a black hole. To extend our instanton – black hole dictionary we investigate the behaviour of the five-dimensional metric at the centers and interpret it in terms of four-dimensional quantities.

Line elements of the form

$$ds_{(5)}^2 = -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} \delta_{mn} dx^m dx^n$$

²⁰For the special case of the dilaton-axion system, this was also observed in [38].

describe extremal black holes with the horizon located at $r = 0$ if the function $e^{-\tilde{\sigma}}$ has the asymptotics

$$e^{-\tilde{\sigma}} \approx \frac{Z}{r^2},$$

where Z is constant. Here we use a spherical coordinate system which is centered at the black hole horizon. The asymptotic line element

$$ds_{(5)}^2 = -\frac{r^4}{Z^2} dt^2 + \frac{Z}{r^2} dr^2 + Z d\Omega_{(3)}^2$$

is locally isometric to $AdS^2 \times S^3$, and the area A of the event horizon is given by the area $A = 2\pi^2 Z^{3/2}$ of the asymptotic three-sphere located at $r = 0$.

To obtain a four-dimensional interpretation, we consider a hypersurface of constant time. The resulting four-dimensional line element

$$ds_{(4)}^2 = e^{-\tilde{\sigma}} \delta_{mn} dx^m dx^n$$

describes the instanton in a conformal frame which is different from the (four-dimensional) Einstein frame employed so far. We will call this frame the Kaluza-Klein frame, and refer to [38] for a more detailed discussion of its role and properties. By definition, the four-dimensional Kaluza-Klein metric is the pull back of the five-dimensional metric onto a hypersurface $t = \text{const.}$, i.e. a constant time hypersurface of the black hole space-time. In this frame the instanton line element is not flat, but only conformally flat. The geometry can be interpreted as a semi-infinite wormhole, which is asymptotically flat for $r \rightarrow \infty$ and ends with a neck of size proportional to the area A of the black hole for $r \rightarrow 0$. For multi-centered solutions there are several such throats with asymptotic sizes given by the areas of the corresponding horizons.

If the constant Z vanishes, the area of the black hole and the neck of the corresponding wormhole have zero size. As is well known from supergravity solutions²¹, a non-vanishing Z requires ‘to switch on sufficiently many charges’. A more precise statement will be made later when we consider explicit examples. Solutions with $Z = 0$ can be interpreted as degenerate black hole solutions with vanishing area of the event horizon. In this case the horizon coincides with the curvature singularity, and the space-time has a null singularity. The spatial geometry corresponds to a semi-infinite wormhole with zero-sized neck. One expects that a finite horizon is obtained when taking into account higher curvature corrections to the Einstein-Hilbert term [58]. Such black holes are called small black holes, in contrast to large black holes which already have a finite horizon at the two-derivative level.

6.4 Attractor behaviour and examples

We will now consider some explicit examples for illustration. Then we return to the general case and show that the asymptotic behaviour at the event horizons is governed by an attractor mechanism which generalizes the one of five-dimensional supergravity.

²¹See for example [57].

6.4.1 Prepotential $\hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \sigma^3$

We start with the STU-type prepotential $\hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \sigma^3$. Like all models with a homogeneous cubic prepotential this model is supersymmetric, or more precisely, a subsector of a supersymmetric model [38]. The dual coordinates are $\sigma_I \simeq \partial_I \log(\sigma^1 \sigma^2 \sigma^3) \simeq \frac{1}{\sigma^I}$, and for convenience we fix the normalization to

$$\sigma_I = \frac{1}{\sigma^I}.$$

Then the four-dimensional instanton solution is given by

$$\sigma^I(x) = \frac{1}{H_I(x)},$$

where $x \in \mathbb{R}^4$. The Kaluza-Klein scalar $\tilde{\sigma}$ is

$$e^{3\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \sigma^3 = \frac{1}{H_1 H_2 H_3}.$$

The resulting five-dimensional line element is

$$\begin{aligned} ds_{(5)}^2 &= -e^{2\tilde{\sigma}} dt^2 + e^{-\tilde{\sigma}} \delta_{mn} dx^m dx^n \\ &= -(H_1 H_2 H_3)^{-2/3} dt^2 + (H_1 H_2 H_3)^{1/3} \delta_{mn} dx^m dx^n, \end{aligned}$$

which is the standard form of a (single or multi-centered) five-dimensional BPS black hole for an STU-model. Observe that the asymptotic metric at the centers is $AdS^2 \times S^3$ if all three harmonic functions are non-constant. This requires to have three non-vanishing charges q_1, q_2, q_3 . If one or two charges are switched off, one obtains ‘small’ black holes with vanishing horizon area.

The result for the five-dimensional scalars h^I is:

$$h^I = e^{-\tilde{\sigma}} \sigma^I = \left(\frac{H_J H_K}{H_I^2} \right)^{1/3},$$

where I, J, K are pairwise distinct. Observe that the h^I take finite fixed point values at the centers, which only depend on the charges. For concreteness, single-centered harmonic functions $H_I = h_I + \frac{q_I}{r^2}$ give

$$h^I \xrightarrow{r \rightarrow 0} \left(\frac{q_J q_K}{q_I^2} \right)^{1/3}.$$

A particular subclass is provided by double-extremal black holes, where the scalars h^I are constant. The fixed point behaviour implies that these constant values are not arbitrary, but determined by the charges. For double-extremal black holes the harmonic functions H_I must be proportional to one another, and the line element takes the form

$$ds_{(5)}^2 = -H^{-2}(x) dt^2 + H(x) \delta_{mn} dx^m dx^n, \quad (6.21)$$

where $H(x)$ is a harmonic function. This is the Tanghelini solution, which is the five-dimensional version of the extremal Reissner-Nordström solution [59].

Models with a general homogeneous cubic prepotential can be treated in an analogous way. However, in general it is not possible to find explicit expressions for the scalars h^I or σ^I in terms of the harmonic functions.

6.4.2 Prepotential $\hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \sigma^3 \sigma^4$

Let us also consider one example which does not correspond to a supersymmetric model. We take the simplest example $\hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \sigma^3 \sigma^4$ of a homogeneous quartic prepotential. We normalize the dual scalars such that

$$\sigma_I = \frac{1}{\sigma^I} ,$$

so that the solution is given by

$$\sigma^I(x) = \frac{1}{H_I(x)} .$$

The corresponding Kaluza-Klein scalar $\tilde{\sigma}$ is

$$e^{4\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma) = \sigma^1 \sigma^2 \sigma^3 \sigma^4 = \frac{1}{H_1 H_2 H_3 H_4} ,$$

which leads to a five-dimensional line element of the form

$$ds_{(5)}^2 = -(H_1 H_2 H_3 H_4)^{-2} dt^2 + (H_1 H_2 H_3 H_4)^4 \delta_{mn} dx^m dx^n .$$

Multi-centered black hole solutions with finite horizons are thus obtained if all four harmonic functions are non-constant, i.e. one needs four non-vanishing charges q_1, \dots, q_4 . The solution for the five-dimensional scalars is

$$h^I = e^{-\tilde{\sigma}} \sigma^I = \left(\frac{H_J H_K H_L}{H_I^3} \right)^{1/4} ,$$

where I, J, K, L are pairwise distinct. Again we observe attractor behaviour, as the five-dimensional scalars approach fixed point values at the centers which only depend on the charges. For a single-centered solution we find

$$h^I \xrightarrow{r \rightarrow 0} \left(\frac{q_J q_K q_L}{q_I^3} \right)^{1/4} .$$

If the scalars are frozen to their fixed point values we obtain a double extreme solution with a Tanghelini line element (6.21).

6.4.3 General homogeneous prepotential $\hat{\mathcal{V}}(\sigma)$

We now return to the general case and consider a general homogeneous prepotential. The dual coordinates

$$\sigma_I \simeq \frac{\hat{\mathcal{V}}_I}{\hat{\mathcal{V}}}$$

are homogenous functions of degree -1 . The solution is given by $\sigma_I(x) = H_I(x)$, where $H_I(x)$ are harmonic functions. While we cannot solve this for the scalars σ^I in closed form, we know that the dual scalars behave like $\sigma_I \sim \frac{1}{r^2}$ at the centers, which implies $\sigma^I \sim r^2$. The asymptotics of the metric at the centers is determined by

$$e^{-\tilde{\sigma}} = \hat{\mathcal{V}}^{-1/p} \approx \frac{Z}{r^2} ,$$

and a finite event horizon requires finite Z . This imposes constraints on the charges, which we discuss below.

If we express the relation $\sigma_I = H_I$ in terms of five-dimensional quantities we obtain

$$e^{-\tilde{\sigma}} \frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} = H_I$$

This has the same form as the generalized stabilisation equations of five-dimensional supergravity [12] and should be interpreted as a generalisation thereof. The generalized stabilisation equations are the algebraic version of the first order flow equations which determine the black hole solution globally.

The stabilization or attractor equations which determine the behaviour at the centers can be obtained by taking the limit $r \rightarrow 0$. In this limit we have

$$H_I \approx \frac{q_I}{r^2}, \quad e^{-\tilde{\sigma}} \approx \frac{Z}{r^2}.$$

The limit $r \rightarrow 0$ of the generalized stabilisation equation gives

$$Z \left. \frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} \right|_* = q_I$$

where $*$ denotes the evaluation at the horizon. This has the same form as the stabilisation equations (attractor equations) of five-dimensional supergravity [12] and should be interpreted as a generalisation thereof. Since

$$\frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} h^I = p \hat{\mathcal{V}}(h) = p,$$

the constant Z can be expressed as

$$Z = \frac{1}{p} q_I h_*^I.$$

Thus the area of the event horizon of the black hole and the size of the neck of the corresponding wormhole/instanton are determined by Z through the charges q_I and the attractor values of the scalars h^I . For supersymmetric models ($p = 3$), Z is proportional to the five-dimensional central charge.

We can be more specific about the conditions leading to a non-vanishing Z if we restrict the functional form of $\hat{\mathcal{V}}(\sigma)$. Consider the case where $\hat{\mathcal{V}}(\sigma)$ is a homogeneous polynomial of degree $p > 0$,

$$\hat{\mathcal{V}}(\sigma) = C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}.$$

Then the dual fields have the form

$$\sigma_I \simeq \frac{\partial_I C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}}{C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}}.$$

Two extremal situations can arise. If the prepotential has the form

$$\hat{\mathcal{V}}(\sigma) = \sigma^1 \dots \sigma^p,$$

then the solution is given by

$$\hat{\mathcal{V}} = (H_1 \cdots H_p)^{-1}$$

and

$$e^{-\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma)^{-1/p} = (H_1 \cdots H_p)^{1/p}.$$

In this case a finite horizon requires that all charges are switched on, i.e. $q_1 \neq 0, \dots, q_p \neq 0$. The other extreme case is a prepotential of the form $\hat{\mathcal{V}} = \sigma^p$. Then the solution is given by

$$\hat{\mathcal{V}} = H^{-p}$$

and

$$e^{-\tilde{\sigma}} = \hat{\mathcal{V}}(\sigma)^{-1/p} = H.$$

In this case it is sufficient to switch on one single charge, because the corresponding scalar enters into the prepotential with the p -th power. General homogeneous prepotentials provide examples for all kinds of cases which lie between these two extreme cases.

The results of this section generalize the results on five-dimensional BPS black holes to a much larger class of non-supersymmetric models defined by homogeneous prepotentials. We observe that for the attractor mechanism the five-dimensional scalars h^I are more suitable, since they have finite fixed point values while the four-dimensional scalars σ^I go to zero. However, since h^I and σ^I are related by a rescaling, both descriptions are equivalent, and the asymptotic fixed point of infinity of the σ^I corresponds to the proper fixed point for the h^I .

7. Conclusions and Outlook

In this paper we have constructed multi-centered extremal black hole solutions using temporal reduction without imposing spherical symmetry. By imposing that the solution can be expressed algebraically in terms of harmonic functions, we have identified a class of scalar geometries which is characterized by the existence of (Hesse or para-Kähler) potential for the metric. This class of theories contains supergravity theories as a subset while preserving the salient feature of BPS solutions, namely multi-centered generalizations and the generalized stabilisation equations. The distinction between BPS and non-BPS extremal solutions in supergravity has been subsumed under the geometrical distinction between solutions which flow along eigendistributions of the para-complex structure and those which flow along other completely isotropic submanifolds of the (extended) scalar manifold. Starting from the interpretation of the equation of motion as defining a harmonic map between the (reduced) space-time and the (extended) scalar manifold, the solution can be expressed algebraically in terms of harmonic functions without the need to bring the equations of motion to first order form. A first order rewriting can still be obtained by imposing that the solution carries finite charges. We plan to use this link to explore the relation between our formalism and the approaches using first order rewritings, ‘fake’-supersymmetry and Hamilton-Jacobi theory. It should also be interesting to investigate how Hessian scalar manifolds could be used within the entropy function formalism of Sen [60]. This approach

allows to study the near horizon geometry of generic Einstein-Maxwell type theories, but it is in general not possible to learn much about the extension of solutions away from the horizon. For BPS solutions one can make the transition from near-horizon to global solutions because generalized stabilisation equations and the ‘proper’ stabilisation equations have the same structure, and we have seen that this feature generalizes to a large class of non-supersymmetric theories. The electric BPS-solutions of five-dimensional are a subclass of our solutions, and one expects that the corresponding instantons are BPS solutions of the reduced four-dimensional Euclidean theory. This can indeed be verified directly, and in [48] we will give a more detailed account on instanton solutions for Euclidean vector multiplets.

For concreteness, we have restricted ourselves in this paper to the relation between five-dimensional Einstein-Maxwell theories and four-dimensional Euclidean sigma models, and to extremal and electro-static backgrounds. This leaves various directions for future work. Evidently, many features of our constructions will generalize to any number of dimensions, the most interesting pair being four-dimensional Einstein-Maxwell type theories and three-dimensional sigma models. Moreover, there are various other types of solutions, like black holes in anti-de-Sitter and de-Sitter space, rotating black holes, black strings and black rings, Taub-NUT spaces, solutions including higher curvature terms, and non-extremal solutions. While some of these might just correspond to more complicated harmonic maps, others will require generalizations of the set up, since the temporal reduction will in general lead to Euclidean theories which also contain gauge fields and a scalar potential. It will be interesting to see whether solutions can be constructed efficiently in such a generalized set up. In this respect it is encouraging that black ring solutions for five-dimensional Einstein-Maxwell-Dilaton gravity have been constructed by lifting solutions of four-dimensional Euclidean sigma models with (symmetric) para-complex target spaces [61].

Besides the construction and study of solutions, the geometrical structures underlying the Lagrangians are very interesting. Both [39] and our work suggest that there are natural generalizations of the special geometries realized in supersymmetric theories. Besides the existence of a potential, homogeneity conditions play an important role, which indicates that the underlying manifolds have homothetic Killing vector fields. This is well known feature of the scalar geometries of vector, tensor and hypermultiplets when these are considered in the superconformal formalism.

As mentioned at various places in this paper, the scalar geometries of theories obtained from the same higher-dimensional theory by dimensional reduction over space and time, respectively, are related by analytic continuation. We have also noticed that this is related to an ambiguity in singling out ‘the’ Euclidean action of a given theory. This has been discussed in some detail in [38], the role of these ambiguities for instanton effects is currently under investigation [47]. Here we would like to point out that these ambiguities suggest to work in the framework of complex-Riemannian geometry and to regard scalar manifolds which are related by analytic continuation as real forms of a single underlying manifold. Interestingly, similar analytic continuations, the complexification of field space, and the subsequent classification of reality conditions seem to play an important role in recent studies of black holes, instantons, domain walls and cosmological solutions within

the framework of ‘fake’-supersymmetry, see for example [29, 51]. While so far such investigations have been restricted to symmetric target spaces, complex-Riemannian geometry should provide the appropriate framework for extending these studies to general targets. Some elements needed for this are provided in the appendix.

A. Complexification of the target space

At the end of Section 2.1 we observed that the target spaces M and M' of the two Euclidean actions (2.1) and (2.5) (equivalently (2.7) and (2.6)) can be viewed as real sections of one underlying complex manifold. Complexification of the action is used in some approaches to defining the Euclidean actions of supersymmetric theories [40]. Complex actions for the ten-dimensional and eleven-dimensional supergravity theories were discussed in [51], while [38] found that a similar formalism should be useful for Euclidean vector multiplets in four dimensions.

Since the scalar target spaces of (2.5) and (2.1) already have a complex or para-complex structure, respectively, before we complexify them some care is needed in order to distinguish between the different complex structures.²² In the following we work out some details and arrive at the conclusion that the proper geometrical framework for complexified Euclidean actions is complex-Riemannian geometry.

When we use the real coordinates (σ^I, b^I) , then the metrics of the target spaces M and M' , which underly the actions (2.1) and (2.5) have the form

$$ds^2 = N_{IJ}(\sigma)(d\sigma^I d\sigma^J \mp db^I db^J) ,$$

respectively. The two line elements are related by the analytic continuation $b^I \rightarrow ib^I$. If we complexify the b^I , then M and M' can be viewed as subspaces of a $3n$ -dimensional space \tilde{M} .

This description is not satisfactory for various reasons. Complexifying only the b^I introduces an asymmetry between the σ^I and the b^I . It is more natural to complexify all fields and to view M and M' as real forms of a complex space M_c . Moreover, M and M' carry additional structures. M is a complex space, and when we use complex coordinates $Y^I = \sigma^I + ib^I$ the line element of M' is manifestly Hermitian

$$ds_{M'}^2 = N_{IJ}(Y + \bar{Y})dY^I d\bar{Y}^J .$$

If we want to view M' as a real form of a complex space M_c , then we need to be careful in distinguishing the complex structure of M' , and the complex structure of M_c which is introduced in the process of complexification, and which is used in the analytic continuation from M' to M . Similarly M is a para-complex space and when using the para-complex coordinates $X^I = \sigma^I + eb^I$, the line element of M is manifestly para-Hermitian:

$$ds_M^2 = N_{IJ}(X + \bar{X})dX^I d\bar{X}^J .$$

²²If one includes fermions then yet another complex structure becomes relevant, namely the one carried by the spinor representation [36]. Here we will restrict ourselves to bosonic actions.

In the following²³ we will reserve the symbol i for the imaginary unit associated with the complex structure of M , while the imaginary unit associated with the complex structure of M_c will be denoted j . We can define j in terms of i and e by observing that the analytic continuation from M' to M , when written in para-complex coordinates, takes the form

$$Y^I = \sigma^I + ib^I \rightarrow X^I = \sigma^I + eb^I .$$

The replacement $ib^I \rightarrow eb^I$ is induced by $b^I \rightarrow (-ie)b^I$, and implies that j should be defined as $j = -ie$. To have $j^2 = -1$ we need to impose the relation $ie = ei$. These relations are consistent and define a four-dimensional commutative and associative real algebra, with basis $1, i, e, j$. This algebra is generated by i and e , subject to the relations

$$i^2 = -1 , \quad e^2 = 1 , \quad ie = ei , \tag{A.1}$$

and defining $j = -ie$. Equivalently, this algebra is generated by i and j subject to the relations

$$i^2 = -1 , \quad j^2 = -1 , \quad ij = ji$$

and defining $e = ij$. The second presentation shows that the algebra is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. Note that this is not only an algebra over \mathbb{R} but also an algebra over \mathbb{C} .

To see that the complex algebra $\mathbb{C} \oplus \mathbb{C}$ is the ‘complexification of the complex numbers’ \mathbb{C} as well as the ‘complexification of the para-complex numbers’ C , recall that the complexification of a real algebra A (associative, with unit) is obtained by taking the real tensor product (of algebras) with \mathbb{C} (considered as a real algebra):

$$A_c = \mathbb{C} \otimes_{\mathbb{R}} A .$$

If one takes $A = \mathbb{C}$ then the result is

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C} . \tag{A.2}$$

If one takes A to be the para-complex numbers C , one obtains the same result:

$$\mathbb{C} \otimes_{\mathbb{R}} C \simeq \mathbb{C} \oplus \mathbb{C} \tag{A.3}$$

The isomorphisms (A.2) and (A.3) can easily be written down explicitly in terms of bases. Alternatively we can simply note that \mathbb{C} and C are real Clifford algebras:

$$Cl_{1,0} \simeq \mathbb{C} , \quad Cl_{0,1} \simeq \mathbb{R} \oplus \mathbb{R} \simeq C ,$$

which is obvious from the relations $i^2 = -1$ and $e^2 = 1$ of the generating elements i and e . Given this, we can refer to the known fact that the complexifications of these two Clifford algebras are [52, 53]:

$$Cl_1 = \mathbb{C} \otimes_{\mathbb{R}} Cl_{1,0} = \mathbb{C} \otimes_{\mathbb{R}} Cl_{0,1} \simeq \mathbb{C} \oplus \mathbb{C} .$$

²³The mathematical background material relevant for the following paragraphs can be found in [52, 53] and [36, 38].

For models with $2n$ free scalar fields the target spaces are simply (the affine spaces associated to the) vector spaces $M = C^n$ and $M' = \mathbb{C}^n$. Both are real $2n$ -dimensional vector spaces, but carry additional structures: $M' = \mathbb{C}^n$ is a (complex-) n -dimensional vector space, $M = C^n$ is an n -dimensional free module over the algebra of para-complex numbers C [36, 38]. Since the complexifications of the underlying algebras C and \mathbb{C} coincide, the complexifications of M and M' are also isomorphic:

$$M_c \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}^n \simeq \mathbb{C} \otimes_{\mathbb{R}} C^n \simeq \mathbb{C}^n \oplus \mathbb{C}^n \simeq \mathbb{C}^{2n} .$$

For models with $2n$ interacting fields, the target spaces M and M' are (para-)complex manifolds, i.e. manifolds modelled on C^n and \mathbb{C}^n , respectively. Both are in particular $2n$ -dimensional real manifolds, and the complexification M_c is a (complex-) $2n$ -dimensional complex manifold. The target spaces of M, M', M_c are $T_P M = C^n$, $T_P M' = \mathbb{C}^n$ and $T_P M_c = \mathbb{C}^{2n} \simeq \mathbb{C} \otimes_{\mathbb{R}} C^n \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}^n$, respectively.

The dynamics of the scalar fields is controlled by the (pseudo-)Riemannian metrics of M and M' . To study the effect of complexification, let us start with the case of 2 free real scalar fields σ and b . Then the real, positive definite line element of M' is

$$ds_{M'}^2 = d\sigma d\sigma + db db .$$

This can be complexified by promoting the real fields σ, b to complex fields:

$$\begin{aligned} \sigma &\rightarrow \Sigma = \sigma_1 + j\sigma_2 , \\ b &\rightarrow B = b_1 + jb_2 . \end{aligned}$$

Here j is the imaginary unit associated with the complex structure of M_c . The resulting complex line element is

$$ds_{M_c}^2 = d\Sigma d\Sigma + dB dB = [d\sigma_1 d\sigma_1 - d\sigma_2 d\sigma_2 + db_1 db_1 - db_2 db_2] + 2j[d\sigma_1 d\sigma_2 + db_1 db_2] .$$

The line element of M' is recovered by taking the real section $\sigma_2 = b_2 = 0$. If we take instead the real section $\sigma_2 = b_1 = 0$ we obtain the real line element

$$ds^2 = d\sigma_1 d\sigma_1 - db_2 db_2 ,$$

which has split signature. Upon setting $\sigma_1 = \sigma$ and $b = b_2$ we obtain the line element

$$ds_M^2 = d\sigma d\sigma - db db$$

of M . Conversely, if we complexify ds_M^2 by (A.4), then we obtain complex line element

$$ds_{M_c}^2 = d\Sigma d\Sigma - dB dB = [d\sigma_1 d\sigma_1 - d\sigma_2 d\sigma_2 - db_1 db_1 + db_2 db_2] + 2j[d\sigma_1 d\sigma_2 - db_1 db_2] .$$

The line elements $ds_{M_c}^2$ and $ds_{M_c'}^2$ are related by the $B \rightarrow jB$, and define the same complex metric on M_c .

The complexification can also be formulated in terms of the complex field $Y = \sigma + ib$. Here the distinction between i and j is important to avoid confusion. In terms of Y , the line element of M is

$$ds_M^2 = dY d\bar{Y} .$$

Here ‘complexification of Y ’ can be understood as ‘taking Y and \bar{Y} to be independent complex variables’. Using the distinction between i and j we can make this precise:

$$\begin{aligned} Y &= \sigma + ib \rightarrow \Sigma + iB , \\ \bar{Y} &= \sigma - ib \rightarrow \Sigma - iB , \end{aligned}$$

with complex fields $\Sigma = \sigma_1 + j\sigma_1$ and $B = b_1 + jb_2$. Similarly, the complex line element of M_c can be obtained by ‘complexifying the para-complex field’ $X = \sigma + eb$.

The most general case we are interested in are line elements of the form

$$ds_{M/M'}^2 = N_{IJ}(\sigma)(d\sigma^I d\sigma^J \pm db^I db^J) .$$

The increase in the number of fields does not change much, as we only need to introduce indices I, J to label the fields. If the real metric $N_{IJ}(\sigma)$ is not flat, we need to assume that it is real-analytic in the σ^I , so that it can be extended analytically to a holomorphic matrix function $N_{IJ}(\Sigma)$, in some neighbourhood of $\sigma_2^I = 0$. The resulting complex manifold M_c contains M and M' as the real submanifolds $\sigma_2 = b_2 = 0$ and $\sigma_2 = b_1 = 0$, respectively. For the purpose of embedding M and M' into some complex manifold, it is not relevant how we choose the neighbourhood of $\sigma_2 = 0$. The resulting line element

$$ds_{M_c}^2 = N_{IJ}(\Sigma)(d\Sigma^I d\Sigma^J + dB^I dB^J)$$

defines a complex-Riemannian metric on M_c . A complex-Riemannian metric on a complex manifold is a complex bilinear form on the holomorphic tangent bundle.²⁴ Note that Kähler (more generally Hermitean) manifolds are Riemannian manifolds and not complex-Riemannian manifolds: they carry a positive definite hermitean sesquilinear form on their (complexified) tangent bundle, whose real part is a Riemannian metric. Similarly para-Kähler (and pseudo-Kähler, para-Hermitean, pseudo-Hermitean) manifolds are pseudo-Riemannian, not complex-Riemannian. The n real shift isometries $b^I \rightarrow b^I + c^I$ induce n complex shift isometries $B^I \rightarrow B^I + C^I$ on M_c .

Symmetric spaces (which are listed in [55] and [56]) provide plenty of examples for triples (M, M', M_c) . The simplest example is

$$M \simeq \frac{SL_2(\mathbb{R})}{SO(1,1)} , \quad M' \simeq \frac{SL_2(\mathbb{R})}{SO(2)} , \quad M_c \simeq \frac{SL_2(\mathbb{C})}{GL(1, \mathbb{C})} .$$

The space $\frac{SL(2, \mathbb{R})}{SO(1,1)}$ occurred in several examples in the main paper. In Section 4 we have seen explicitly that this pseudo-Riemannian symmetric is para-Kähler, and that it is related by analytic continuation to the Riemannian symmetric space $\frac{SL(2, \mathbb{R})}{SO(2)}$, which is Kähler. Note that while the above example uses symmetric spaces, the discussion in this appendix applies to analytic (pseudo-)Riemannian manifolds in general.

²⁴A definition of complex-Riemannian manifolds and some further references can be found in [54]. The extension of the bilinear form to the anti-holomorphic tangent bundle is given by complex conjugation. Taking the holomorphic and anti-holomorphic tangent bundles to be orthogonal, one obtains a natural extension to the full (complexified) tangent bundle.

Acknowledgements

We would like to thank Gabriel Lopes Cardoso and Vicente Cortés for various valuable discussion. T.M. thanks the Center for Mathematical Physics of the University of Hamburg for support and hospitality during several visits to Hamburg. He also thanks the LMU Munich for hospitality and the Royal Society for support of this work through the Joint Projects Grant ‘Black Holes, Instantons and String Vacua’.

References

- [1] K. Stelle, *BPS Branes in Supergravity*, hep-th/9803116.
- [2] S.D. Majumdar, *A class of exact solutions of Einstein’s field equations*, Phys. Rev. 72 (1947) 390.
A. Papapetrou, Proc. Roy. Irish Acad. A51 (1947) 191.
- [3] G.W. Gibbons, *Supersymmetric Soliton States in Extended Supergravity Theories*. In P. Breitenlohner and H.P. Dürr (eds.), *Unified Theories of Elementary Particles*, Springer, 1982.
- [4] G.W. Gibbons and C.M. Hull, *A Bogomolny Bound for General Relativity and Solitons in $N = 2$ Supergravity*, Phys. Lett. B 106 (1982) 190.
- [5] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ Extremal Black Holes*, Phys. Rev. D 52 (1995) 5412, hep-th/9508072.
- [6] A. Strominger, *Macroscopic Entropy of $N = 2$ Extremal Black Holes*, Phys. Lett. D 383 (1996) 39, hep-th/9602111.
- [7] S. Ferrara and R. Kallosh, *Supersymmetry and Attractors*, Phys. Rev. D 54 (1996) 1514, hep-th/9602136, *Universality of Supersymmetric Attractors*, Phys. Rev. D 54 (1996) 1525, hep-th/9603090.
- [8] K. Behrndt, G.L. Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, *Classical and Quantum $N = 2$ Supersymmetric Black Holes*, Nucl. Phys. B 488 (1997) 236, hep-th/9610105.
- [9] W. Sabra, *General Static $N = 2$ Black Holes*, Mod. Phys. Lett. A 12 (1997) 2585, hep-th/9703101, *Black Holes in $N = 2$ Supergravity and Harmonic Functions*, Nucl. Phys. B 510 (1998) 247, hep-th/9704147.
- [10] K. Behrndt, D. Lüst and W.A. Sabra, *Stationary Solutions of $N = 2$ Supergravity*, Nucl. Phys. B 510 (1998) 264, hep-th/9705169.
- [11] W.A. Sabra, *General BPS Black Holes in Five Dimensions*, Mod. Phys. Lett. A 13 (1998) 239, hep-th/9708103.
- [12] A.H. Chamseddine and W.A. Sabra, *Metrics Admitting Killing Spinors in Five Dimensions*, Phys. Lett. B 426 (1998) 36, hep-th/9801161.
- [13] G.L. Cardoso, B. de Wit, J. Käppeli and T. Mohaupt, *Stationary BPS Solutions in $N = 2$ Supergravity with R^2 interactions* JHEP 12 (2000) 019, hep-th/0009234.
- [14] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. B 500 (1997) 75, hep-th/9702103.
- [15] G.W. Moore, *Arithmetic and Attractors*, hep-th/9807087.
- [16] F. Denef, *Supergravity Flows and D-brane Stability*, JHEP 08 (2000) 050, hep-th/0005049.

- [17] K. Goldstein, N. Iizuka, R.P. Jena and S.P. Trivedi, *Non-supersymmetric Attractors*, Phys. Rev. D 72 (2005) 124021, hep-th/0507096.
- [18] P.K. Tripathi and S.P. Trivedi, *Non-supersymmetric Attractors in String Theory*, JHEP 03 (2006) 022, hep-th/0511117.
- [19] R. Kallosh, *New Attractors*, JHEP 12 (2005) 022, hep-th/0510024, R. Kallosh, N. Sivanandam and M. Soroush, *The non-BPS Black Hole Attractor Equation*, JHEP 03 (2006) 060, hep-th/0602005, *Exact Attractive non-BPS STU Black Holes*, Phys. Rev. D 74 (2006) 065008, hep-th/0606263.
- [20] S. Ferrara, K. Hayakawa and A. Marrani, *Erice Lectures on Black Holes and Attractors*, Fortschr. Phys. 56 (2008) 993, arXiv:0805.2498.
- [21] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, Lect. Notes in Physics 737 (2008) 661, hep-th/0611345.
- [22] A. Ceresole and G. Dall'Agata, *Flow Equations for non-BPS Extremal Black Holes*, JHEP 03 (2007) 110, hep-th/0702088.
- [23] G.L. Cardoso, A. Ceresole, G. Dall'Agata, J.M. Oberreuter and J. Perz, *First Order Flow Equations for Extremal Black Holes in Very Special Geometry*, JHEP 10 (2007) 063, arXiv:0706.3373.
- [24] J. Perz, P. Smyth, T. Van Riet and B. Vercnocke, *First-order flow equations for extremal and non-extremal black holes*, JHEP 03 (2009) 150, arXiv:0810.1528.
- [25] M. Chiodaroli and M. Gutperle, *Instantons and Wormholes in $N = 2$ Supergravity*, arXiv:0901.1616.
- [26] W. Chemissany, J. Roessel, M. Trigiante and T. Van Riet, *The Full Integration of Black Hole Solutions to Symmetric Supergravity Theories*, arXiv:0903.2777.
- [27] L. Andrianopoli, R. D'Auria, E. Orazi and M. Trigiante, *First Order Description of $D = 4$ Static Black Holes and the Hamilton-Jacobi Equation*, arXiv:0905.3938.
- [28] D.Z. Freedman, C. Nunez, M. Schnabl and K. Skenderis, *Fake Supergravity and Domain Wall Stability*, Phys. Rev. D 69 (2004) 104027, hep-th/0312055.
A. Celi, A. Ceresole, G. Dall'Agata, A. Van Proeyen and M. Zagermann, *On the Fakeness of Fake Supergravity*, Phys. Rev. D 71 (2005) 045009, hep-th/0410126.
K. Skenderis and P.K. Townsend, *Pseudo-supersymmetry and the Domain Wall/Cosmology Correspondence*, J. Phys. A 40 (2007) 6733, hep-th/0610253.
- [29] P.K. Townsend, *From Wave Geometry to Fake Supergravity*, J. Phys. A 41 (2008) 304014, arXiv:0710.5709.
- [30] G. Neugebauer and D. Kramer, *Eine Methode zur Konstruktion stationärer Einstein-Maxwell-Felder*, Ann. der Physik (Leipzig) 24 (1969) 253.
- [31] P. Breitenlohner, D. Maison and G. Gibbons, *Four-dimensional black holes from Kaluza-Klein theories*, Comm. Math. Phys. 120 (1988) 253.
- [32] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante and T. Van Riet, *Generating Geodesic Flows and Supergravity Solutions*, Nucl. Phys. B 812 (2009) 343, arXiv: 0806.2310.
- [33] M. Gunaydin, A. Neitzke, B. Pioline and A. Waldron, *BPS black holes, quantum attractor flows and automorphic forms*, Phys. Rev. D 73 (2006) 084019, hep-th/0512296, *Quantum attractor flows*, JHEP 09 (2007) 056, arXiv:0707.0267.

- [34] D. Gaiotto, W. Li and M. Padi, *Non-Supersymmetric Attractor Flow in Symmetric Spaces*, JHEP 12 (2007) 093, arXiv:0710.1638.
W. Li, *Non-Supersymmetric Attractors in Symmetric Coset Spaces*, arXiv:0801.2536.
- [35] M. Gunaydin, G. Sierra and P.K. Townsend, *The Geometry of $N = 2$ Maxwell Einstein Supergravity and Jordan Algebras*, Nucl. Phys. B 242 (1984) 244.
- [36] V. Cortés, C. Mayer, T. Mohaupt, and F. Saueressig, *Special Geometry of Euclidean Supersymmetry I: Vector Multiplets*, JHEP 03, (2004) 028, hep-th/0312001.
- [37] V. Cortés, C. Mayer, T. Mohaupt, and F. Saueressig, *Special Geometry of Euclidean Supersymmetry II: Hypermultiplets and the c -map*, JHEP 09 (2005) 025, hep-th/0503094.
- [38] V. Cortés and T. Mohaupt, *Special Geometry of Euclidean Supersymmetry III: the local r -map, instantons and black holes*, JHEP 07 (2009) 066, arXiv:0905.2844.
- [39] D. Alekseevsky and V. Cortés, *Geometric Construction of the r -map: From Affine Special Real to Special Kähler Manifolds*, arXiv:0811.1658.
- [40] P. van Nieuwenhuizen and A. Waldron, *On Euclidean Spinors and Wick Rotations*, Phys. Lett. B 389 (1996) 29, hep-th/9608174, *A Continuous Wick Rotation for Spinor Fields and Supersymmetry in Euclidean Space*, hep-th/9611043.
- [41] G.W. Gibbons, M.B. Green and M. Perry, *Instantons and Seven-Branes in Type IIB Superstring*, Phys. Lett. B 370 (1996) 37, hep-th/9511080. M.B. Green and M. Gutperle, *Effects of D -instantons*, Nucl. Phys. B 498 (1997) 195, hep-th/9701093.
- [42] K. Behrndt, I. Gaida, D. Lüst, S. Mahapatra and T. Mohaupt, *From Type IIA Black Holes to T -dual Type IIB D -instantons in $N = 2$, $D = 4$ Supergravity*, Nucl. Phys. B 508 (1997) 659, hep-th/9706096.
- [43] M. Gutperle and M. Spalinski, *Supergravity Instantons and the Universal Hypermultiplet*, JHEP 06 (2000) 037, hep-th/0005068, *Supergravity Instantons for $N = 2$ Hypermultiplets*, Nucl. Phys. B 598 (2001) 509, hep-th/0010192.
- [44] U. Theis and S. Vandoren, *Instantons in the Double-Tensor Multiplet*, JHEP 09 (2002) 059, hep-th/0208145.
- [45] M. Davidse, M. de Vroome, U. Theis and S. Vandoren, *Instanton Solutions for the Universal Hypermultiplet*, Fortschr. Phys. 52 (2004) 696, hep-th/0309220.
- [46] M. de Vroome and S. Vandoren, *Supergravity Description of Spacetime Instantons*, Class. Quant. Grav. 24 (2007) 509, hep-th/0607055.
- [47] T. Mohaupt and K. Waite, work in progress.
- [48] T. Mohaupt and K. Waite, in preparation.
- [49] G. Clement and D. Gal'tsov, *Stationary BPS solutions to dilaton-axion gravity*, Phys. Rev. D 54 (1996) 6136, hep-th/9607043.
D. Gal'tsov and O.A. Rychkov, *Generating branes via sigma-models*, Phys. Rev. D 58 (1998) 122001, hep-th/9801160.
- [50] B. de Wit and A. Van Proeyen, *Special Geometry, Cubic Polynomials and Homogeneous Quaternionic Spaces*, Comm. Math. Phys. 149 (1992) 307.
- [51] E. A. Bergshoeff, J. Hartong A. Ploegh, J. Rosseel and D. Van den Bleeken, *Pseudo-supersymmetry and a Tale of Alternate Realities*, JHEP 07 (2007) 067, arXiv:0704.3559.

- [52] P. Lounesto, *Clifford Algebras and Spinors*, Cambridge UP 2001.
- [53] H.B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton UP, 1989.
- [54] S. Ivanov, *Holomorphic Projective Transformations on Complex Riemannian Manifold*, Jour. of Geom. 49 (1994) 106.
- [55] R. Gilmore, *Lie Groups, Lie Algebras and Some of Their Applications*, Wiley Interscience, 1974.
- [56] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, 1978.
- [57] J. Maldacena, *Black Holes in String Theory*, PhD Thesis, Harvard, 1996, hep-th/9607235.
- [58] A. Dabholkar, R. Kallosh, and A. Maloney, *A Stringy Cloak for a Classical Singularity*, JHEP 12 (2004) 059, hep-th/0410076.
- [59] F.R. Tangherlini, *Schwarzschild Field in n Dimensions and the Dimensionality of Space Problem*, Nuovo Cimento 27 (1963) 636.
- [60] A. Sen, *Black Hole Entropy Function, Attractors, and Precision Counting of Microstates*, Gen. Rel. Grav. 40 (2008) 2249, arXiv:0708.1270.
- [61] S.S. Yazadjiev, *Generating Dyonic Solutions in 5D Einstein-dilaton gravity with Antisymmetric Forms and Dyonic Black Rings*, Phys. Rev. D 73 (2006) 124032, hep-th/0512229, *Completely Integrable Sector in 5D Einstein-Maxwell Gravity and Derivation of the Dipole Black Ring Solutions*, Phys. Rev. D 73 (2006) 104007, hep-th/0602116, *Solution Generating in 5D Einstein-Maxwell-Dilaton Gravity and Derivation of Dipole Black Ring Solutions*, JHEP 07 (2006) 036, hep-th/0604140.